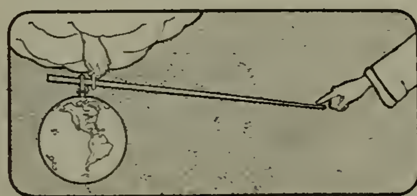
THE SOLAR PARALLAX

AND ITS

RELATED CONSTANTS

HARKNESS



WASHINGTON OBSERVATIONS FOR 1885.—APPENDIX III.

THE SOLAR PARALLAX AND ITS RELATED CONSTANTS, INCLUDING THE FIGURE AND DENSITY OF THE EARTH.

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ON THE SOLAR PARALLAX AND ITS RELATED CONSTANTS.*

1.—INTRODUCTORY.

Hitherto it has been customary to endeavor to determine the solar parallax as if it were an independent constant, and the result is a mass of discordant values, all of which are more or less affected by constant errors, and none of which command anything like universal assent. But, in truth, the solar parallax is not an independent constant. On the contrary, it is entangled with the lunar parallax, the constants of precession and nutation, the parallactic inequality of the Moon, the lunar inequality of the Earth, the masses of the Earth and Moon, the ratio of the solar and lunar tides, the constant of aberration, the velocity of light, and the light equation; and according to the most elementary mathematical principles, it should be determined simultaneously with all these quantities, by means of a least square adjustment. No other method offers anything like so much promise of eliminating the ever present constant errors, and therefore an attempt will be made to develop it here. The equations connecting the quantities mentioned are known, but for the sake of completeness their derivation will be given. The theory of these equations and the discussion of the numerical quantities which they involve are entirely distinct subjects, and as clearness will be gained by separating them, we will begin by investigating the numerical values, both of the constants, and of the quantities to be adjusted.

2.—ALGEBRAIC NOTATION, AND CITATION OF AUTHORITIES.

Except where otherwise stated, the following notation will be employed in algebraic formulæ:

a = equatorial semi-axis of the globe of the Earth, if that body is regarded as a spheroid; or major equatorial semi-axis, if it is regarded as an ellipsoid.

a' = minor equatorial semi-axis of the Earth, when that body is regarded as an ellipsoid.

b = polar semi-axis of the Earth.

a_1 = that distance between the Earth and the Sun which would satisfy KEPLER'S third law.

* In its original form this paper was a private investigation, and as such was read before the Philosophical Society of Washington on October 13, 1888. Since then it has been taken up in connection with my official work, to which it is closely related, and many important details have been added.—W. H.

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a_2 = that distance between the Earth and the Moon which would satisfy KEPLER'S third law.
E = the combined mass of the Earth and Moon.
E′ = the mass of the Earth, excluding the Moon.
e = eccentricity of the globe of the Earth.
e_1 = eccentricity of the Earth's orbit.
e_2 = eccentricity of the Moon's orbit.
g = observed force of gravity at a specified point upon the Earth's surface.
I = inclination of the Moon's orbit to the plane of the ecliptic.
k = GAUSS's constant for the solar system.
L = constant of the Earth's lunar inequality.
l = length of a simple pendulum vibrating in the time t.
M = mass of the Moon.
m = ratio of the mean motions of the Sun and Moon.
$\mathfrak{N}$ = the constant of nutation.
$\mathfrak{P}$ = the constant of luni-solar precession.
P = the constant of lunar parallax.
p = the constant of solar parallax.
Q = the parallactic inequality of the Moon.
r = that value of the mean distance from the Earth to the Sun which is adopted in the solar tables.
r_1 = that value of the mean distance from the Earth to the Moon which is adopted in the lunar tables.
S = mass of the Sun.
s' = geocentric latitude of the Moon.
T = length of the sidereal year, expressed in seconds of mean time.
T_1 = length of the sidereal month, expressed in seconds of mean time.
t = time.
t_1 = number of mean solar seconds in a sidereal day.
V = the velocity of light per second of mean time.
α = the constant of aberration.
γ = DELAUNAY's constant, which is sin $\frac{1}{2}$ (inclination of lunar orbit to plane of ecliptic).
ε = $(a - b)/a$ = the quantity variously designated as the ellipticity, compression, or flattening of the Earth.
θ = the time taken by light to traverse the mean radius of the Earth's orbit.
$\varkappa$ = a constant, such that $a_1 = r\ (1 + \varkappa)$
$\varkappa'$ = a constant, such that $a_2 = r_1\ (1 + \varkappa')$
μ = regression of Moon's node, relatively to the line of equinoxes, in $365\frac{1}{4}$ days.
ν = the heliocentric longitude of the Earth.
ν' = the geocentric longitude of the Moon.
ρ = a factor, varying with the latitude, such that the radius of the globe of the Earth at latitude φ is $a\rho$.
σ = centrifugal force at latitude φ, divided by gravity at the same latitude.

φ = geographic latitude.
φ' = geocentric latitude.
ψ = the luni-solar precession.
ψ_1 = the general precession.
ω_0 = the mean obliquity of the fixed ecliptic at the initial epoch.
ω = the obliquity of the fixed ecliptic at the time t.
ω_1 = the obliquity of the moving ecliptic at the time t.

For a list of the principal works consulted in the preparation of the present paper, and an explanation of the method by which they are cited in the foot-notes, the reader is referred to the section on bibliography, pages 146–165.

3.—RELATIONS BETWEEN UNITS OF LENGTH.

For interchanging the various units of length employed by different authors, the following ratios, based upon Gen. A. R. Clarke's determinations,* will be used throughout this paper:

		Logarithms.
1 meter	= 3·280 869 33 feet	0·515 988 934
1 kilometer	= 0·621 376 767 mile	9·793 355 011
1 toise	= 6·394 533 48 feet	0·805 808 865
1 toise	= 1·949 036 318 meters	0·289 819 931
1 Paris line	= 0·002 255 829 072 meter	7·353 306 189
1 statute mile	= 1·609 329 561 kilometers	0·206 644 989
1 English inch	= 0·025 399 772 meter	8·404 829 820

4.—SIZE AND FIGURE OF THE EARTH.

When the Earth is regarded as a spheroid, let a be its equatorial and b its polar semi-axis; and when it is regarded as an ellipsoid, let a and a' be respectively its major and minor equatorial, and b its polar semi-axis. Then the flattening will be given by the formula

$$\varepsilon = \frac{a-b}{a} \tag{1}$$

and the ratio of the axes will be

$$a:b = \frac{1}{\varepsilon} : \frac{1}{\varepsilon} - 1 \tag{2}$$

In 1799 Méchain and Delambre found from a combination of the arc of 9° 40′ which they had measured between Dunkerque and Barcelona with the arc of 3° 07′ measured in Peru by Bouguer and La Condamine†

$$\text{Log. } a \text{ (in meters)} = 6{\cdot}804\ 5305\ 074$$
$$\text{Log. } b \text{ (in meters)} = 6{\cdot}803\ 2282\ 744$$

Whence

$$a = 6\ 375\ 738{\cdot}66 \text{ meters} = 20\ 917\ 965 \text{ feet}$$
$$b = 6\ 356\ 649{\cdot}63 \text{ meters} = 20\ 855\ 337 \text{ feet}$$
$$\varepsilon = 1/334$$

*13, p. 157, and 23, p. 280.

†25, T. 3, pp. 196 and 432.

In 1830, from fourteen meridional arcs having an amplitude of 59° 29′, and four arcs of parallel having an amplitude of 22° 41′, Sir (then Professor) GEORGE B. AIRY found *

$$a = 20\,923\,713 \text{ feet}$$
$$b = 20\,853\,810 \text{ feet}$$
$$\varepsilon = 1/299{\cdot}3249$$

In 1841, from ten meridional arcs having a total amplitude of 50° 36′, to which he applied a more rigorous analysis than had before been used, BESSEL found †

$$a = 3\,272\,077{\cdot}14 \text{ toises} = 20\,923\,407 \text{ feet}$$
$$b = 3\,261\,139{\cdot}33 \text{ toises} = 20\,853\,465 \text{ feet}$$
$$\varepsilon = 1/299{\cdot}1529$$

In 1858, from a discussion of eight meridional arcs having a total amplitude of 67° 08′, General (then Captain) A. R. CLARKE found, when the curvature of the meridians was not restricted to an elliptic form ‡

$$a = 20\,927\,197 \pm 385 \text{ feet}$$
$$b = 20\,855\,493 \pm 257 \text{ feet}$$
$$\varepsilon = 1/291{\cdot}8554$$

And when the curvature was restricted to an elliptic form §

$$a = 20\,926\,348 \pm 186 \text{ feet}$$
$$b = 20\,855\,233 \pm 239 \text{ feet}$$
$$\varepsilon = 1/294{\cdot}2607$$

In 1859 General T. F. DE SCHUBERT advanced the idea that the well-known discordances between the astronomical and geodetic differences of latitude and longitude of points upon the Earth's surface arise mainly from the assumption that the Earth is a spheroid, when in truth it is an ellipsoid; and, in accordance with that hypothesis, he found from eight meridional arcs having a total amplitude of 72° 37′ ‖

$$a = 3\,272\,671{\cdot}5 \text{ toises} = 20\,927\,207 \text{ feet}$$
$$a' = 3\,272\,303{\cdot}2 \text{ toises} = 20\,924\,852 \text{ feet}$$
$$b = 3\,261\,467{\cdot}9 \text{ toises} = 20\,855\,566 \text{ feet}$$

In 1860 General A. R. CLARKE repeated General DE SCHUBERT'S investigation by applying a much more exact analysis to five meridional arcs having a total amplitude of 76° 35′. He found ¶

$$a = 20\,926\,485 \text{ feet}$$
$$a' = 20\,921\,177 \text{ feet}$$
$$b = 20\,853\,768 \pm 953 \text{ feet}$$

* 17, p. 220. † 19, p. 116. ‡ 21, p. 765. § 21, p. 771. ‖ 26, p. 31. ¶ 22, p. 39.

In 1866 the comparisons made at Southampton showed that the hitherto accepted relations of the principal standards of length were slightly erroneous, and to correct the error thence arising General CLARKE recomputed the axes last given, and found*

$$a = 20\,926\,350 \text{ feet}$$
$$a' = 20\,919\,972 \text{ feet}$$
$$b = 20\,853\,429 \text{ feet}$$

By modifying his equations so as to make them represent a spheroid, he found from the same data†

$$a = 20\,926\,062 \text{ feet}$$
$$b = 20\,855\,121 \text{ feet}$$
$$\varepsilon = 1/294{\cdot}9784$$

In 1878 the serious uncertainty respecting the unit of length employed by Colonel LAMBTON in the measurement of the southern half of the Indian meridional arc had been remedied by a complete remeasurement of that part of the triangulation; the latitudes of many stations in it had been determined; the length of the arc had been increased from 21° 21′ to 23° 50′; and an arc of longitude extending through 10° 28′ had been measured. The data available for determining the size and figure of the Earth were then the Russian arc of 25° 20′, the Anglo-French arc of 22° 10′, the Indian meridional arc of 23° 50′, the Indian longitudinal arc of 10° 28′, the Cape arc of 4° 37′, and the Peruvian arc of 3° 07′. From these six arcs, having a total amplitude of 89° 32′, General (then Colonel) CLARKE found‡

$$a = 20\,926\,629 \text{ feet}$$
$$a' = 20\,925\,105 \text{ feet}$$
$$b = 20\,854\,477 \text{ feet}$$

In 1880, after considering the ellipsoidal theory, and calling special attention to the fact that sufficient data are not yet available for fixing definitively the form of the Earth, General CLARKE reverted to the theory of a spheroid, and found from the arcs he had employed in 1878§

$$a = 20\,926\,202 \text{ feet} \qquad b = 20\,854\,895 \text{ feet} \tag{3}$$

These last values will be adopted in the present paper. From them we have

$$\varepsilon = \frac{a-b}{a} = \frac{1}{293{\cdot}4663} \tag{4}$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 0{\cdot}006\,803\,481\,019 \tag{5}$$

$$\tan \varphi' = \frac{b^2}{a^2} \tan \varphi = (1 - e^2) \tan \varphi = 0{\cdot}993\,196\,519 \tan \varphi \tag{6}$$

$$\rho^2 = \frac{1 - e^2\,(2 - e^2) \sin^2 \varphi}{1 - e^2 \sin^2 \varphi} = \frac{\cos \varphi}{\cos \varphi' \cos (\varphi - \varphi')} \tag{7}$$

*23, p. 285. †23, p. 287. ‡24, p. 92. §13, p. 319.

And with $\sin \varphi = \sqrt{\frac{1}{3}}$

$$\varphi = 35^\circ \; 15' \; 51{\cdot}79'' \qquad \varphi' = 35^\circ \; 04' \; 48{\cdot}76'' \qquad \rho^2 = 0{\cdot}997\,742\,482 \qquad (8)$$

5.—LENGTH OF THE SECONDS PENDULUM.

Let l be the length of the simple pendulum which beats seconds of mean solar time when vibrating through an infinitely small arc, in vacuo, at the level of the sea, in latitude φ. For determining the value of l the following data are available:

In 1799, from observations made at 15 stations, whose latitudes ranged from $+67^\circ \; 05'$ to $-33^\circ \; 56'$, the whole combined in such a way as to make the absolute length of the pendulum depend solely upon BORDA's measurements at the Paris Observatory, LA PLACE found*

$$l = 0{\cdot}739\,502 + 0{\cdot}004\,208 \sin^2 \varphi \text{ meter}$$

But this is the length of the decimal pendulum making 100 000 vibrations in a mean solar day. To reduce it to the sexagesimal pendulum making 86 400 vibrations, it must be multiplied by

$$\left\{ \frac{1000}{864} \right\}^2 = 1{\cdot}339\,591\,906$$

and then it becomes

$$l = 0{\cdot}990\,631 + 0{\cdot}005\,637 \sin^2 \varphi \text{ meter}$$

In 1816, from observations made at 31 stations, whose latitudes ranged from $+74^\circ \; 53'$ to $-51^\circ \; 21'$, MATHIEU found, for the decimal pendulum,†

$$l = 0{\cdot}739\,586 + 0{\cdot}004\,080\,6 \sin^2 \varphi \text{ meter}$$

which becomes for the sexagesimal pendulum

$$l = 0{\cdot}990\,743 + 0{\cdot}005\,466 \sin^2 \varphi \text{ meter}$$

In 1821, from observations at 8 stations, situated in France and Scotland, between latitudes $+38^\circ \; 40'$ and $+60^\circ \; 45'$, BIOT found, for the decimal pendulum,‡

$$l = 0{\cdot}739\,687\,686 + 0{\cdot}003\,986\,392 \sin^2 \varphi \text{ meter}$$

which becomes for the sexagesimal pendulum

$$l = 0{\cdot}990\,880 + 0{\cdot}005\,340 \sin^2 \varphi \text{ meter}$$

In 1825, from his own observations at 13 stations having latitudes ranging from $+79^\circ \; 50'$ to $-12^\circ \; 59'$, Captain (afterwards General Sir) EDWARD SABINE found §

$$\begin{aligned} l &= 39\,015\,68 + 0{\cdot}202\,13 \sin^2 \varphi \text{ English inches} \\ &= 0{\cdot}990\,989 + 0{\cdot}005\,134 \sin^2 \varphi \text{ meter} \end{aligned}$$

* 7, T. 2, livre 3, chap. 5, sec. 42. † 32, p. 332. ‡ 20, p. 575. § 35, p. 352.

And from a combination of his own stations, those of the British survey, and those of the French arc; in all 25 stations, having latitudes ranging from $+79°\ 50'$ to $-12°\ 59'$, he found

$$l = 39.015\,20 \quad + 0.202\,45 \quad \sin^2 \varphi \text{ inches}$$
$$= \quad 0.990\,977 + 0.005\,142 \sin^2 \varphi \text{ meter}$$

In 1827, from observations made at 41 stations, whose latitudes ranged from $+79°\ 50'$ to $-51°\ 35'$, M. SAIGEY found*

$$l = 0.991\,026 + 0.005\,072 \sin^2 \varphi \text{ meter}$$

In 1829, from observations made at 5 stations, whose latitudes ranged from $0°$ to $+67°\ 04'$, PONTÉCOULANT found†

$$l = 0.990\,555 + 0.005\,679 \sin^2 \varphi \text{ meter}$$

In 1830, from observations made at 49 stations, whose latitudes ranged from $+79°\ 50'$ to $-51°\ 35'$, Sir (then Professor) GEORGE B. AIRY found‡

$$l = 39.016\,77 \quad + 0.200\,27 \quad \sin^2 \varphi \text{ inches}$$
$$= \quad 0.991\,017 + 0.005\,087 \sin^2 \varphi \text{ meter}$$

In 1833, by a process which is not clearly described, but which seems to have consisted in making the absolute length of the pendulum depend upon the experiments of BORDA at the Paris Observatory, while the coefficient of $\cos 2\varphi$ was made to depend upon the compression of the Earth, assumed to be $1/288$, POISSON found§

$$l = 0.993\,512\ (1 - 0.002\,588 \cos 2\varphi) \text{ meter}$$

To pass from the form

$$l = a\,(1 - b \cos 2\varphi)$$

to the form

$$l = \alpha + \beta \sin^2 \varphi$$

we have

$$\alpha = a\,(1 - b) \qquad\qquad \beta = 2ab$$

and therefore POISSON'S formula is equivalent to

$$l = 0.990\,941 + 0.005\,142 \sin^2 \varphi \text{ meter}$$

In 1869, from observations made at 51 stations, whose latitudes ranged from $+79°\ 50'$ to $-51°\ 35'$, UNFERDINGER found‖

$$\text{Log. } l \text{ (in Paris lines)} = 2.642\,756\,8 + [7.351\,47] \sin^2 \varphi + [5.319\,8] \sin^4 \varphi + [3.457] \sin^6 \varphi$$

*38, p. 36. †10, T. 2, p. 466. ‡17, p. 230. §15, T. 1, p. 367; 34, pp. 32–33, and 16, T. 2, p. 464. ‖39, p. 316.

In order to reduce this to the standard form, let us assume

$$l = 439{\cdot}2923 + x + (2{\cdot}2940 + y)\sin^2\varphi \qquad (9)$$

and then, by making φ successively equal to 0°, 20°, 40°, 60°, and 80°, and comparing the resulting values of l with those given by UNFERDINGER's expression, we shall obtain the observation equations

$$\begin{aligned} 0 &= x + 0{\cdot}0000\,y - 32 \\ 0 &= x + 0{\cdot}1170 \quad - 10 \\ 0 &= x + 0{\cdot}4132 \quad + 12 \\ 0 &= x + 0{\cdot}7500 \quad - 22 \\ 0 &= x + 0{\cdot}9698 \quad - 78 \end{aligned}$$

where the absolute terms are in units of the fourth decimal place. The resulting normal equations are

$$\begin{aligned} 0 &= +5{\cdot}0000\,x + 2{\cdot}2500\,y - 130{\cdot}0000 \\ 0 &= +2{\cdot}2500 \quad + 1{\cdot}6874 \quad - 88{\cdot}3560 \end{aligned}$$

Whence

$$x = +6{\cdot}093\,057 \qquad\qquad y = +44{\cdot}237\,654$$

and by substitution in (9)

$$\begin{aligned} l &= 439{\cdot}292\,9 \quad + 2{\cdot}298\,42 \ \sin^2\varphi \text{ Paris lines} \\ &= \quad 0{\cdot}990\,970 + 0{\cdot}005\,185 \sin^2\varphi \text{ meter} \end{aligned}$$

In 1884, from observations made at 123 stations, whose latitudes ranged from + 79° 50′ to — 62° 56′, HELMERT found*

$$\begin{aligned} l &= 0{\cdot}990\,918\,[1 + (0{\cdot}005\,310 \pm 14)\sin^2\varphi] \text{ meter} \\ &= 0{\cdot}990\,918 \quad + (0{\cdot}005\,262 \pm 14)\sin^2\varphi \text{ meter} \end{aligned}$$

For convenience of reference the preceding results are collected in Table I. In the case of UNFERDINGER's formula, the value of l for latitude 45° was computed from his original logarithmic expression, and not from that in the third column of the table.

TABLE I.—*Formulæ for the Length of the Seconds Pendulum.*

Date.	Author.	l in meters.		l for $\varphi = 45°$
1799	La Place . . .	0·990 631 + 0·005 637 sin² φ		0·993 450
1816	Mathieu	·990 743	5 466	476
1821	Biot	·990 880	5 340	550
1825	Sabine	·990 977	5 142	548
1827	Saigey	·991 026	5 072	562
1829	Pontécoulant . .	·990 555	5 679	395
1830	Airy	·991 017	5 087	560
1833	Poisson	·990 941	5 142	512
1869	Unferdinger . .	·990 970	5 185	554
1884	Helmert. . . .	0·990 918 + 0·005 262 sin² φ		0·993 549

*14, II Teil, p. 241.

A glance at the table shows that among the various authorities the range in the length of the pendulum is nearly three times greater at the equator than at latitude 45° ; and that was to be expected, because most of the observations have been made within the temperate zone. Judging from the data upon which they are based, the formulæ of SABINE, SAIGEY, AIRY, UNFERDINGER, and HELMERT seem the most trustworthy, and their arithmetical mean gives

$$l = 0.993\,554\,6 \text{ meter at lat. } 45° \tag{10}$$

which is probably not in error by so much as 0·01 of a millimeter.

General CLARKE's proof that the ellipticity of the Earth resulting from pendulum observations does not differ sensibly from that obtained from terrestrial measurements,* would be more satisfactory if he had used a more exact value of the ratio of the force of gravity to the Earth's centrifugal force, but as his contention seems likely to prove correct in the end, we will preserve the consistency of our data by deriving the coefficient of $\sin^2 \varphi$ through CLAIRAUT's theorem. The theoretical expression connecting the length of the pendulum in any latitude with the size and figure of the Earth is†

$$l = l_0 \left\{ 1 + \left(\frac{10a}{l_0 t_1^2} - \frac{a-b}{a} \right) \sin^2 \varphi \right\} \tag{11}$$

where l_0 is the length of the pendulum at the equator, and t_1 is the number of mean solar seconds in a sidereal day. Whence, by putting

$$l = l_0 + \text{B} \sin^2 \varphi$$

we have

$$\text{B} = \frac{10a}{t_1^2} - \frac{l_0(a-b)}{a} \tag{12}$$

With $l_0 = 3.251\,169$ feet; $t_1 = 86\,164.10^s$, from (17); and the values of a and b marked (3); formula (12) gives

$$\text{B} = 0.017\,108 \text{ feet} = 0.005\,214\,4 \text{ meter}$$

and by the combination of that result with (10), we obtain definitively

$$\begin{aligned} l &= 0.990\,947 + 0.005\,214 \sin^2 \varphi \text{ meter} \\ &= 3.251\,169 + 0.017\,108 \sin^2 \varphi \text{ feet} \end{aligned} \tag{13}$$

With $\sin^2 \varphi = 1/3$, formula (13) gives

$$l = 3.256\,872 \text{ English feet.} \tag{14}$$

* 13, p. 348. † 10, T. 2, p. 428.

6.—LENGTH OF THE YEAR.

If we put m for the quantity by which the Sun's mean sidereal motion in $365\frac{1}{4}$ days exceeds $360°$, then the length of the sidereal year will be given by the expression

$$T = 365\tfrac{1}{4}\left\{1 - \frac{m}{360°} + \left(\frac{m}{360°}\right)^2 - \text{etc.}\right\} \text{days} \tag{15}$$

According to HANSEN and OLUFSEN,* $m = -22{\cdot}560\,09''$; whence, by formula (15)

$$T = 365{\cdot}256\,358\,192 \text{ days} = 365^d\ 06^h\ 09^m\ 09{\cdot}347\,8^s$$

According to LE VERRIER the mean sidereal motion of the Sun in $365\frac{1}{4}$ days is† $1\,295\,977{\cdot}382\,34'' + 0{\cdot}060\,3''$. Whence, $m = -22{\cdot}557\,36''$; and by formula (15)

$$T = 365{\cdot}256\,357\,422 \text{ days} = 365^d\ 06^h\ 09^m\ 09{\cdot}281\,2^s$$

As the perturbations of the Earth are not taken account of in precisely the same way by HANSEN and OLUFSEN and by LE VERRIER, the resulting values of the Sun's mean sidereal motion given by these two authorities are not rigorously comparable, but the systematic difference is very small. Neglecting it, we take the mean of the two values of T just found, and thus obtain for the length of the sidereal year, expressed in mean solar time

$$T = 365{\cdot}256\,357\,81 \text{ days} = 365^d\ 06^h\ 09^m\ 09{\cdot}314^s = 31\,558\,149{\cdot}314 \text{ seconds} \tag{16}$$

And the number of mean solar seconds in a sidereal day is

$$t_1 = 86\,400^s \times \frac{365{\cdot}256\,357\,8}{366{\cdot}256\,357\,8} = 86\,164\,099\,65^s \tag{17}$$

If we put

$$m + \psi_1 = a + bt \tag{18}$$

where ψ_1 is the secular part of the general precession, the expression for the length of the tropical year will be

$$T' = 365\tfrac{1}{4}^d \left\{\begin{array}{l} +1 - \frac{a}{360°} + \left(\frac{a}{360°}\right)^2 - \text{etc.} \\ -\left(\frac{1}{360°} - \frac{2a}{(360°)^2}\right)bt + \text{etc.} \end{array}\right\} \tag{19}$$

Taking for m the mean of the values given respectively by HANSEN and OLUFSEN, and by LE VERRIER, namely,

$$m = -\tfrac{1}{2}(22{\cdot}560\,09'' + 22{\cdot}557\,36'') = -22{\cdot}558\,72''$$

*40, p. 1. †41, pp. 52 and 98.

and putting

$$\psi_1 = +\ 50{\cdot}236\ 22'' + 0{\cdot}022\ 044''\left(\frac{t-1850}{100}\right)$$

we have

$$m + \psi_1 = +\ 27{\cdot}677\ 50'' + 0{\cdot}022\ 044''\left(\frac{t-1850}{100}\right)$$

Whence, by formula (19),

$$\begin{aligned} \mathrm{T}' &= 365{\cdot}242\ 199\ 853^{\mathrm{d}} - 0{\cdot}000\ 006\ 212\ 4^{\mathrm{d}}\left(\frac{t-1850}{100}\right) \\ &= 365^{\mathrm{d}}\ 05^{\mathrm{h}}\ 48^{\mathrm{m}}\ 46{\cdot}067^{\mathrm{s}} - 0{\cdot}536\ 75^{\mathrm{s}}\left(\frac{t-1850}{100}\right) \end{aligned} \qquad (20)$$

The variation in the length of the sidereal or tropical year produced by a small change in the adopted value of m, formula (15), or a, formula (19), may be readily found from one or other of the expressions

$$\begin{aligned} d\mathrm{T} &= -\ 24{\cdot}35 dm \\ d\mathrm{T}' &= -\ 24{\cdot}35 da \end{aligned} \qquad (21)$$

where $d\mathrm{T}$ and $d\mathrm{T}'$ are in seconds of time, and dm and da in seconds of arc.

7.—THE ECCENTRICITY OF THE EARTH'S ORBIT AND THE CONSTANT $(1 + \nu)$.

In their Tables du Soleil, published in 1853, HANSEN and OLUFSEN gave for the epoch 1850·0,* $e_1 = 0{\cdot}016\ 771\ 20$; but in his Darlegung, published in 1862, HANSEN gave for the epoch 1800·0,† $e_1 = 0{\cdot}016\ 792\ 26$ Using the co-efficients of (23) to bring the latter result up to the epoch 1850·0, it becomes

$$e_1 = 0{\cdot}016\ 771\ 035 \qquad (22)$$

In 1858 LE VERRIER gave‡

$$e_1 = 3\ 459{\cdot}28'' - 0{\cdot}087\ 55''\ (t-1850) - 0{\cdot}000\ 282''\left(\frac{t-1850}{100}\right)^2$$

or, in parts of radius

$$e_1 = 0{\cdot}016\ 771\ 063 - 0{\cdot}000\ 000\ 424\ 5\ (t-1850) - 0{\cdot}000\ 000\ 001\ 367\left(\frac{t-1850}{100}\right)^2 \qquad (23)$$

Adopting the mean of (22) and (23), we have for the epoch 1850.0

$$e_1 = 0{\cdot}016\ 771\ 049 \qquad (24)$$

*40, p. 1. †55, p. 176. ‡41, p. 102.

According to LE VERRIER*

$$r = a_1 \left(1 + \frac{2\sigma}{3n} \right)$$

and as we have put

$$a_1 = r\,(1 + \varkappa)$$

it follows that

$$1 + \varkappa = \frac{1}{1 + \frac{2\sigma}{3n}}$$

Here n is the Earth's mean sidereal motion, and σ is a quantity depending upon the perturbations of that motion by the other planets. For the numerical values of these constants LE VERRIER found†

$$\text{Log. } n = 6{\cdot}112\,60 \qquad \sigma = +\,2{\cdot}506\,7''$$

Whence

$$1 + \frac{2\sigma}{3n} = 1{\cdot}000\,001\,289\,5$$

and

$$1 + \varkappa = 0{\cdot}999\,998\,710 \tag{25}$$

8.—LENGTH OF THE MONTH.

The best authorities for the data from which to derive the length of the month are AIRY, HANSEN, and NEWCOMB.

The following notation will be employed:

g = mean anomaly of the Moon.
g' = mean anomaly of the Sun.
ω = distance of the lunar perigee from the Moon's ascending node.
ω' = distance of the solar perigee from the Moon's ascending node.
Θ = supplement of the longitude of the Moon's ascending node.
ψ_1 = the secular part of the general precession.
χ = sidereal movement of the solar perigee in $365\frac{1}{4}$ days.
n'' = the Moon's mean sidereal motion in $365\frac{1}{4}$ days.
n_1'' = the Moon's mean synodical motion in $365\frac{1}{4}$ days.
n_2'' = the Moon's mean tropical motion in $365\frac{1}{4}$ days.
n_3'' = the Moon's mean nodical motion in $365\frac{1}{4}$ days.
n_4'' = the Moon's mean anomalistic motion in $365\frac{1}{4}$ days.

*8, T. 2, pp. 30, 31, and 60.

†8, T. 2, p. 59, and 41, p. 11.

The mean motion of the Moon used by DAMOISEAU can best be found by taking from his Table II* the mean longitudes for the years — 300 and — 2300 and dividing their difference by 2000. The resulting mean tropical motion in $365\frac{1}{4}$ days is

$$5347{\cdot}420\,869\,125 \text{ centesimal degrees}$$

or, expressed in sexagesimal seconds, of which there are 3240 in each centesimal degree, $17\,325\,643{\cdot}616\,0''$. To that motion AIRY found the correction $+0{\cdot}596''$;† and as he used BESSEL'S value of the precession, which for the epoch 1800·0 is $50{\cdot}2235''$, his value of the mean sidereal motion of the Moon in $365\frac{1}{4}$ days is

$$17\,325\,643{\cdot}616\,0'' - 50{\cdot}223\,5'' + 0{\cdot}596'' = 17\,325\,593{\cdot}988\,5''$$

and adding the secular acceleration‡

$$n'' = 17\,325\,593{\cdot}985\,8'' + 0{\cdot}216\,4''\left(\frac{t - 1800}{100}\right) \tag{26}$$

It is easily seen that

$$n'' = \frac{d}{dt}\left(g + \varpi - \Theta - \psi_1\right) \tag{27}$$

and also

$$n'' = \frac{d}{dt}\left(g + \varpi - \varpi' + \chi\right) \tag{28}$$

But when the numerical values of the quantities in the right hand members of these equations are substituted from HANSEN'S Tables de la Lune, pp. 15 and 16, a discordance appears in the terms involving the square of the time; equation (27) giving for the coefficient $+0{\cdot}000\,332\,60''$, while equation (28) gives $+0{\cdot}000\,404\,19''$. Taking the mean of these two numbers, HANSEN'S value of the mean sidereal motion of the Moon in $365\frac{1}{4}$ days is

$$n'' = 17\,325\,593{\cdot}973\,1'' + 0{\cdot}243\,60''\left(\frac{t - 1800}{100}\right) + 0{\cdot}000\,368\,40''\left(\frac{t - 1800}{100}\right)^2 \tag{29}$$

In his "Researches on the Motion of the Moon," NEWCOMB found that HANSEN'S tables required the correction§

$$-1{\cdot}14'' - 0{\cdot}291\,7''\,(t - 1800) - 3{\cdot}86''\left(\frac{t - 1800}{100}\right)^2$$

*50, p. 3. †44, p. 10. ‡44, p. 8. §60, pp. 268 and 274.

to the mean longitude of the Moon; and the correction $+0{\cdot}10''\,(t-1800)$ to the motion of the Moon's node. Or, in other words, if we put g_0, g'_0, ϖ_0, ϖ'_0 and Θ_0 for NEWCOMB's corrected values of the quantities g, g', ϖ, ϖ', and Θ, then

$$\begin{aligned} g_0 &= g - 1{\cdot}14'' - 0{\cdot}2917''\,(t-1800) - 3{\cdot}86''\left(\frac{t-1800}{100}\right)^2 \\ g'_0 &= g' \\ \varpi_0 &= \varpi - 0{\cdot}10''\,(t-1800) \\ \varpi'_0 &= \varpi' - 0{\cdot}10''\,(t-1800) \\ \Theta_0 &= \Theta - 0{\cdot}10''\,(t-1800) \end{aligned} \qquad (30)$$

and the substitution of these corrected values in equations (27) and (28) gives

$$n'' = \frac{d}{dt}(g+\varpi-\Theta-\psi_1) - 0{\cdot}2917'' - 0{\cdot}0772\left(\frac{t-1800}{100}\right) \qquad (31)$$

$$= \frac{d}{dt}(g+\varpi-\varpi'+\chi) - 0{\cdot}2917'' - 0{\cdot}0772\left(\frac{t-1800}{100}\right) \qquad (32)$$

NEWCOMB's value of the mean sidereal motion of the Moon in $365\frac{1}{4}$ days is therefore

$$\begin{aligned} n'' = 17\,325\,593{\cdot}681\,4'' &+ 0{\cdot}166\,40''\left(\frac{t-1800}{100}\right) \\ &+ 0{\cdot}000\,368\,40''\left(\frac{t-1800}{100}\right)^2 \end{aligned} \qquad (33)$$

If we put

$$n'' = a + bt + ct^2 \qquad (34)$$

the expression for T_1, the length of the sidereal month, will be

$$T_1 = \frac{365\frac{1}{4}^{d} \times 360^\circ}{a+bt+ct^2} = \frac{473\,364\,000''}{a}\left\{1 - \frac{b}{a}t + \left(\frac{b^2}{a^2} - \frac{c}{a}\right)t^2 - \text{etc.}\right\} \qquad (35)$$

to terms of the third order in $(t-1800)$.

Formulæ (26), (29), and (33) are of the same form as (34), and the several values of T_1 are to be determined from them by substituting in (35) the values of a and b which they contain. In view of the uncertainty as to the true value of the secular acceleration of the Moon's motion, the terms depending upon the square of the time can have no real significance, and will be neglected.

From (26) and (35), according to AIRY

$$\begin{aligned} T_1 &= 27{\cdot}321\,660\,682^{d} - 0{\cdot}000\,000\,341\,25^{d}\left(\frac{t-1800}{100}\right) \\ &= 27^{d}\,07^{h}\,43^{m}\,11{\cdot}483^{s} - 0{\cdot}029\,484^{s}\left(\frac{t-1800}{100}\right) \end{aligned} \qquad (36)$$

From (29) and (35), according to HANSEN

$$T_1 = 27{\cdot}321\,660\,702^d - 0{\cdot}000\,000\,384\,15^d\left(\frac{t-1800}{100}\right)$$
$$= 27^d\,07^h\,43^m\,11{\cdot}484^s - 0{\cdot}033\,191^s\left(\frac{t-1800}{100}\right) \qquad (37)$$

And from (33) and (35), according to NEWCOMB

$$T_1 = 27{\cdot}321\,661\,162^d - 0{\cdot}000\,000\,262\,40^d\left(\frac{t-1800}{100}\right)$$
$$= 27^d\,07^h\,43^m\,11{\cdot}524^s - 0{\cdot}022\,671^s\left(\frac{t-1800}{100}\right) \qquad (38)$$

Equations (36), (37), and (38) yield the following values for the mean length of the sidereal month at the epoch 1850·0, expressed in mean solar seconds:

AIRY	2 360 591·468^s	
HANSEN	2 360 591·467	(39)
NEWCOMB	2 360 591·513	

For the mean synodical, tropical, nodical, and anomalistic motions of the Moon we have

$$n_1'' = \frac{d}{dt}(g - g' + \varpi - \varpi') - 0{\cdot}2917'' - 0{\cdot}0772''\left(\frac{t-1800}{100}\right) \qquad (40)$$

$$n_2'' = \frac{d}{dt}(g + \varpi - \Theta) - 0{\cdot}2917'' - 0{\cdot}0772''\left(\frac{t-1800}{100}\right) \qquad (41)$$

$$n_3'' = \frac{d}{dt}(g + \varpi) - 0{\cdot}3917'' - 0{\cdot}0772''\left(\frac{t-1800}{100}\right) \qquad (42)$$

$$n_4'' = \frac{d}{dt}(g) - 0{\cdot}2917'' - 0{\cdot}0772''\left(\frac{t-1800}{100}\right) \qquad (43)$$

where in each case the literal part of the expression is HANSEN's value, and the numerical terms are NEWCOMB's corrections. Of these expressions the most important is (40). HANSEN's value for it is

$$n_1'' = 16\,029\,616{\cdot}5331'' + 0{\cdot}243\,824''\left(\frac{t-1800}{100}\right) + 0{\cdot}000\,404\,19''\left(\frac{t-1800}{100}\right)^2 \qquad (44)$$

and NEWCOMB's value is

$$n_1'' = 16\,029\,616{\cdot}2414'' + 0{\cdot}166\,624''\left(\frac{t-1800}{100}\right) + 0{\cdot}000\,404\,19''\left(\frac{t-1800}{100}\right)^2 \qquad (45)$$

Substituting these values of n_1'' in formula (35), and neglecting the terms involving the square of the time, the mean length of the synodical month comes out, according to HANSEN

$$\begin{aligned} &29{\cdot}530\,587\,898^{d} - 0{\cdot}000\,000\,449\,19^{d}\left(\frac{t-1800}{100}\right) \\ &= 29^{d}\ 12^{h}\ 44^{m}\ 02{\cdot}794^{s} - 0{\cdot}038\,810^{s}\left(\frac{t-1800}{100}\right) \end{aligned} \qquad (46)$$

and according to NEWCOMB

$$\begin{aligned} &29{\cdot}530\,588\,435^{d} - 0{\cdot}000\,000\,306\,96^{d}\left(\frac{t-1800}{100}\right) \\ &= 29^{d}\ 12^{h}\ 44^{m}\ 02{\cdot}841^{s} - 0{\cdot}026\,522^{s}\left(\frac{t-1800}{100}\right) \end{aligned} \qquad (47)$$

As a check on the preceding computations, we may employ the relation

$$T_1 (T + T_2) = TT_2 \qquad (48)$$

where T, T_1 and T_2 are respectively the lengths of the sidereal year, the sidereal month, and the synodical month. With the values of these quantities from (16), (38) and (47), we find

$$\begin{aligned} T_1 (T + T_2) &= 10\,786{\cdot}235\,174\,5 \text{ days} \\ TT_2 &= 10\,786{\cdot}235\,173\,6 \text{ days} \end{aligned}$$

For the other months, at the epoch 1850·0, we have from equations (41), (42), and (43), by substituting the values of g, ω, and Θ from HANSEN's lunar tables, and neglecting NEWCOMB's corrections

$$\begin{aligned} n_2'' &= 17\,325\,644{\cdot}33'' \\ n_3'' &= 17\,395\,273{\cdot}64 \\ n_4'' &= 17\,179\,158{\cdot}87 \end{aligned} \qquad (49)$$

whence

$$\begin{aligned} &\text{Mean Tropical Month} = 27{\cdot}321\,581\,292^{d} = 27^{d}\ 07^{h}\ 43^{m}\ 04{\cdot}624^{s} \\ &\text{Mean Nodical Month} = 27{\cdot}212\,219\,238^{d} = 27^{d}\ 05^{h}\ 05^{m}\ 35{\cdot}742^{s} \\ &\text{Mean Anomalistic Month} = 27{\cdot}554\,550\,463^{d} = 27^{d}\ 13^{h}\ 18^{m}\ 33{\cdot}160^{s} \end{aligned} \qquad (50)$$

9.—THE CONSTANTS μ, m, $(1 + \varkappa')$, e_2, γ, AND I, PERTAINING TO THE MOON.

For the regression of the Moon's node relatively to the line of equinoxes, in $365\frac{1}{4}$ days, HANSEN's lunar tables give *

$$\mu = -\frac{d}{dt}(\Theta + \psi_1) = -69\,679{\cdot}6191'' + 0{\cdot}141\,36''\left(\frac{t-1800}{100}\right) \qquad (51)$$

Whence, for the epoch 1850·0

$$\mu = 19^{\circ}\ 21'\ 19{\cdot}5484'' = 0{\cdot}337\,815\,984 \text{ of radius} \qquad (52)$$

* 54, pp. 15–16.

Or, with Professor NEWCOMB's correction, from (30)

$$\mu = 19^\circ\ 21'\ 19{\cdot}4484'' = 0{\cdot}337\,815\,499 \text{ of radius} \tag{53}$$

The ratio of the mean sidereal motions of the Sun and Moon results immediately from (39) and (16), thus

$$m = \frac{T_1}{T} = \frac{2\,360\,591{\cdot}513^s}{31\,558\,149{\cdot}314^s} = 0{\cdot}074\,801\,329\,112 \tag{54}$$

By putting

$$a_2 = r_1 (1 + \varkappa')$$

and comparing this with the expression given by DELAUNAY for the constant part of the lunar parallax, namely*

$$\frac{1}{r_1} = \frac{1}{a_2} \left\{ 1 + \left(\frac{1}{6} + \frac{1}{4} e_1^2\right) m^2 - \frac{179}{288} m^4 - \frac{97}{48} m^5 \right\} \tag{55}$$

we find

$$1 + \varkappa' = 1 + \left(\frac{1}{6} + \frac{1}{4} e_1^2\right) m^2 - \frac{179}{288} m^4 - \frac{97}{48} m^5 \tag{56}$$

Whence, by substituting the values of e_1 and m from (24) and (54)

$$1 + \varkappa' = 1{\cdot}000\,908\,743 \tag{57}$$

From all the lunar observations made at Greenwich between the years 1750 and 1847, inclusive, Sir G. B. AIRY found for the coefficient of the first term of the equation of the Moon's center†

$$22\,639{\cdot}06'' \tag{58}$$

Professor NEWCOMB found that for the same term HANSEN used in his lunar tables‡

$$22\,640{\cdot}15'' \tag{59}$$

and from all the meridian observations of the Moon made at Greenwich between the years 1847 and 1874, and at Washington between the years 1862 and 1874, inclusive, NEWCOMB found the correction $-0{\cdot}57''$ to (59),§ which thus becomes

$$22\,639{\cdot}58'' \tag{60}$$

Taking the mean of (58) and (60), giving the former double weight, and equating the result to its analytical equivalent,|| we obtain

$$22\,639{\cdot}233'' = 2\,e_2 - \frac{1}{4} e_2^3 + \frac{5}{96} e_2^5.$$

*52, T. 2, p. 914. Compare also 63, T. 1, pp. 664 and 674.

†44, p. 13.

‡62, p. 69.

§61, p. 29, and 62, p. 69.

||52, T. 2, p. 804, equation (7).

Whence

$$e_2 = 11\,323{\cdot}880'' = 0{\cdot}054\,899\,720 \text{ of radius} \quad (61)$$

For the coefficient of the first term in the development of the Moon's latitude, expressed as a function of the Moon's true longitude, AIRY found from the Greenwich observations made between the years 1750 and 1847, inclusive,* 18 535·55″; but when expressed as a function of the time, this becomes, according to DELAUNAY†

$$18\,461{\cdot}26'' \quad (62)$$

Professor NEWCOMB found that the value of the same coefficient implicitly contained in HANSEN's lunar tables is‡ 18 461·629″; and to that NEWCOMB found a correction of −0·15″ from the Greenwich and Washington observations of the Moon made between the years 1862 and 1874.§ His corrected value is therefore

$$18\,461{\cdot}48'' \quad (63)$$

Taking the mean of (62) and (63), giving the former double weight, and equating the result to DELAUNAY's analytical expression for it,‖ we have

$$18\,461{\cdot}33'' = 0{\cdot}089\,503\,054 \text{ of radius}$$

$$= 2\gamma - 2\gamma e_2^2 - \frac{1}{4}\gamma^5 + \frac{7}{32}\gamma e_2^4 + \frac{1}{4}\gamma^5 e_2^2 - \frac{5}{144}\gamma e_2^6$$

whence, with the value of e_2 from (61)

$$\gamma = 0{\cdot}044\,886\,793 \quad (64)$$

As γ is the sine of half the inclination of the Moon's orbit to the plane of the ecliptic, (64) gives

$$\text{I} = 5^\circ\ 08'\ 43{\cdot}3546'' = 0{\cdot}089\,803\,757 \text{ of radius.} \quad (65)$$

10.—OBSERVED VALUE OF THE PARALLACTIC INEQUALITY OF THE MOON.

From the nature of the case, the observations for determining the coefficient of this inequality must be made partly upon the first, and partly upon the second limb of the Moon, and thus they involve all the systematic errors which may arise from the different conditions under which these limbs are observed, and all the uncertainty which attaches to our knowledge of the Moon's semi-diameter. The following values are perhaps the best hitherto obtained:

1. From his discussion of the lunar observations made at Greenwich between the years 1811 and 1851, Sir G. B. AIRY concluded that the most probable value of this coefficient is 124·7″¶

2. Professor NEWCOMB found that the value deduced by HANSEN from the Greenwich and Dorpat observations is 126·46″ **

* 44, p. 21, and 45, p. 27.
† 52, T. 2, p, 802.
‡ 62, p. 76.
§ 61, p. 36.
‖ 52, T. 2, p. 862, equation (1).
¶ 44, p. 16.
** 232, p. 23.

3. From 2 075 lunar observations made at Greenwich between the years 1848 and 1866, Mr. E. J. STONE found 125·36″ ± 0·4″; the probable error being estimated.*

4. From the lunar observations made at Washington between the years 1862 and 1866, Professor NEWCOMB found 125·46″.†

5. From an extended discussion of the whole subject, Messrs. CAMPBELL and NEISON found, either 125·64″ ± 0·09″, or 124·64″ ± 0·25″, according as a certain hypothetical 45-year term was or was not admitted into the lunar theory.‡

6. From a comparison of HANSEN'S lunar tables with some 1 600 observations of the Moon, made with the Greenwich transit circle between the years 1862 and 1877, Mr. EDMUND NEISON found 125·313″ ± 0·046″; but he thought it probable that that value might require diminution by 0·73″ on account of the before-mentioned hypothetical 45-year term.§

The controversy carried on in the Monthly Notices of the Royal Astronomical Society by Mr. STONE and Messrs. CAMPBELL and NEISON, during the years 1880–'82, shows that the entire mass of existing lunar observations must be thoroughly rediscussed before a definitive value of the parallactic coefficient can be obtained. Respecting the data given above, it may be remarked that the values in paragraph 5 are superseded by that in paragraph 6, and that the mean of the values in paragraphs 1 and 2 is nearly the same as the values in paragraphs 3, 4, and 6. All questions of weights may therefore be disregarded, and by taking the arithmetical mean of all the values except those in paragraph 5, we find

$$Q = 125{\cdot}46'' \pm 0{\cdot}35'' \tag{66}$$

where the probable error is estimated to be one-fifth of the difference between the greatest and least values.

11.—OBSERVED VALUE OF THE LUNAR INEQUALITY OF THE EARTH.

The magnitude of the coefficient of this inequality is only about two-thirds that of the solar parallax, but as it depends upon differences of the sun's right ascension, which is always observed in precisely the same way, it should be free from constant errors, and can therefore be determined with great accuracy. The following are the best available data:

From observations at Greenwich, Paris, and Kœnigsberg, made during the periods stated, LE VERRIER found ‖

Greenwich	1816–1826	L = 6·45″
Greenwich	1827–1850	6·56
Paris	1804–1814	6·61
Paris	1815–1845	6·47
Kœnigsberg	1814–1830	6·43
		Mean = 6·50 ± 0·023″

*65, p. 271. †232, p. 24. ‡46, p. 467. §59, p. 409. ‖41, p. 100.

Professor NEWCOMB found the following additional values*

Greenwich	1851–1864	$L = 6.56'' \pm 0.04''$
Washington	1861–1865	6.51 ± 0.07

Giving these three results weights inversely proportional to the squares of their probable errors, namely, 9·26, 3·06, and 1, we obtain

$$L = 6.514'' \pm 0.016'' \tag{67}$$

12.—OBSERVED VALUE OF THE LUNAR PARALLAX.

For the determination of this constant, all the data at present available are based upon declinations of the Moon, observed respectively in Europe and at the Cape of Good Hope.

From a comparison of LACAILLE's observations at the Cape with those made at Greenwich, Paris, Berlin, and Bologna, during the same period, namely, from June, 1751, to February, 1753, Professor OLUFSEN found†

$$x = \sin P = 0.016\,512\,33 + 0.024\,492\,01\,\varepsilon - 0.000\,001\,62\,dL$$

or, multiplying by $3423.3''/\sin 3423.3'' = 206\,274.28$

$$P = 3406.069'' + 5052.072''\,\varepsilon - 0.334''\,dL \pm 0.45'' \tag{68}$$

where ε is the Earth's compression, and dL the error in the longitude of the Cape, expressed in minutes of time. Neglecting dL, we have from (4)

$$\varepsilon = \frac{1}{293.466}$$

whence

$$P = 3406.069'' + 17.215'' = 3423.284'' \pm 0.45'' \tag{69}$$

From his own observations of the Moon at the Cape, combined with observations made at Greenwich and Cambridge during the same period, namely, from May, 1832, to May, 1833, Professor THOMAS HENDERSON deduced two values of the lunar parallax; one by comparison with BURCKHARDT's tables, and the other by comparison with DAMOISEAU's tables. As BURCKHARDT's parallaxes are now known to be erroneous, we have only to consider the result from DAMOISEAU's tables, which was‡

$$P = 3422.46'' + 5062''\,\delta c - 0.05''\,\delta t - 0.12\,\delta s - 0.14\,\delta s' \tag{70}$$

where

$$\text{Longitude of Cape Observatory} = 1^h\ 13^m\ 55^s + \delta t^s$$

$$\delta c = \varepsilon - \frac{1}{300}$$

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*232, pp. 25–26. †74, p. 226. ‡73, p. 294.

and δs and $\delta s'$ are corrections for any constant differences which may have existed in the values of the Moon's semi-diameter given by the different instruments, and the different observers.

Neglecting δs and $\delta s'$, we have

$$\delta t = -0.35^s$$

$$\delta e = \frac{1}{293.466} - \frac{1}{300} = 0.000\,074\,22$$

Whence, from (70)

$$P = 3422.46'' + 0.376'' + 0.018'' = 3422.854'' \qquad (71)$$

By combining 123 observations of the Moon made at the Cape during the years 1830 to 1837, with corresponding observations made at Greenwich, Edinburgh, and Cambridge, Mr. BREEN found*

$$P = 3422.696'' - 0.013\,\delta t \qquad (72)$$

where δt is the correction to the assumed longitude of the Cape, and the compression of the Earth is taken to be 1 / 300. With our value of the compression, namely, 1 / 293.466, and $\delta t = -0.35^s$, (72) gives

$$P = 3422.696'' + 0.376'' + 0.005'' = 3423.077'' \qquad (73)$$

By combining 239 observations of the Moon made at the Cape during the years 1856 to 1861 with corresponding observations made at Greenwich, Mr. E. J. STONE found†

$$P = 3422.707'' \pm 0.049''$$

Mr. STONE does not state what compression he employed, but as 1 / 300 was then used both at Greenwich and the Cape, the same correction should probably be applied as in (71) and (73). That gives

$$P = 3422.707'' + 0.376'' = 3423.083'' \pm 0.049'' \qquad (74)$$

Collecting our results, we have from (69), (71), (73) and (74),

P = 3423.284″	according to OLUFSEN.
3422.854	according to HENDERSON.
3423.077	according to BREEN.
3423.083	according to STONE.

When it is remembered that the first of these values is based upon observations made with old-fashioned quadrants, the second and third upon observations made with mural circles, and the fourth upon observations made with large transit circles, their agreement is remarkable. However, as the observations upon which the second result rests have all been embodied in the third, we will adopt the mean of the last two, and put

$$P = 3423.08'' \pm 0.050'' \qquad (75)$$

If we assume the probable errors of the latitudes of Greenwich and the Cape to be $\pm 0.05''$, then the probable error of P should be increased to $\pm 0.121''$

*72, p. 137. †75, p. 16.

13.—THE CONSTANT OF PRECESSION.

Putting ψ and ψ_1 respectively for the secular parts of the luni-solar, and of the general precession; ω for the mean inclination of the equator of $1850 + t$ to the fixed ecliptic of 1850; and ω_1 for the mean inclination of the equator of $1850 + t$ to the ecliptic of $1850 + t$; then, according to LE VERRIER and SERRET, we shall have*

$$\begin{aligned} \psi &= \left\{ \begin{matrix} 50{\cdot}371\,40'' + x + 0{\cdot}014\,5''\,\nu \;\; + 0{\cdot}174\,3''\,\nu^{i} \\ + 0{\cdot}016\,9''\,\nu^{iii} - 0{\cdot}057\,5''\,\nu^{iv} - 0{\cdot}012\,4''\,\nu^{v} \end{matrix} \right\} t - 0{\cdot}000\,108\,806''\,t^2 \\ \omega &= \omega_0 + 0{\cdot}000\,007\,180''\,t^2 \\ \psi_1 &= (50{\cdot}235\,72'' + x)\,t + 0{\cdot}000\,112\,900\,t^2 \\ \omega_1 &= \omega_0 - \left\{ \begin{matrix} 0{\cdot}475\,66'' + 0{\cdot}005\,3''\,\nu + 0{\cdot}288\,8''\,\nu^{i} \\ + 0{\cdot}008\,3''\,\nu^{iii} + 0{\cdot}160\,1''\,\nu^{iv} + 0{\cdot}013\,1''\,\nu^{v} \end{matrix} \right\} t - 0{\cdot}000\,001\,490''\,t^2 \end{aligned} \tag{76}$$

where x is a small unknown correction to the precession constant, and t is counted in Julian years from the epoch 1850·0. From the adopted masses of the planets, given on pages 42 and 48, we have

$$\nu = -0{\cdot}655\,353 \qquad \nu^{iii} = -0{\cdot}133\,558 \qquad \nu^{v} = +0{\cdot}002\,970$$
$$\nu^{i} = -0{\cdot}007\,004 \qquad \nu^{iv} = +0{\cdot}002\,339$$

Whence, by substitution in (76)

$$\begin{aligned} \psi &= (50{\cdot}358\,25'' + x)\,t - 0{\cdot}000\,108\,806''\,t^2 \\ \omega &= \omega_0 + 0{\cdot}000\,007\,180''\,t^2 \\ \psi_1 &= (50{\cdot}235\,72'' + x)\,t + 0{\cdot}000\,112\,900''\,t^2 \\ \omega_1 &= \omega_0 - 0{\cdot}469\,47''\,t - 0{\cdot}000\,001\,490''\,t^2 \end{aligned} \tag{77}$$

From the series of observed values of the obliquity of the ecliptic given by LE VERRIER,† and with the theoretical value just found for the annual change of the obliquity, we deduce for 1850·0

$$\omega_0 = 23^\circ\ 27'\ 31{\cdot}36'' \pm 0{\cdot}345'' \tag{78}$$

To find the planetary precession, λ, we have

$$\psi - \psi_1 = \lambda \cos \tfrac{1}{2} (\omega + \omega_1) \tag{79}$$

where it will be sufficiently accurate to take

$$\cos \tfrac{1}{2}(\omega + \omega_1) = \cos (\omega_0 - 0{\cdot}234\,735''\,t) = \cos \omega_0 + 0{\cdot}234\,735''\,t \text{ arc } 1'' \sin \omega_0 \tag{80}$$

The substitution in (79) of the values of ψ, ψ_1, ω and ω_1, from (77), (78) and (80), gives

$$0{\cdot}122\,53''\,t - 0{\cdot}000\,221\,706''\,t^2 = \lambda\,(0{\cdot}917\,347\,2 + 0{\cdot}000\,000\,453\,t)$$

Whence

$$\lambda = 0{\cdot}133\,570''\,t - 0{\cdot}000\,241\,748''\,t^2 \tag{81}$$

*8, I. 2, p. 174, and 83, p. 324.

†41, p. 51.

Equations (77), (78) and (81) give

$$
\begin{aligned}
\frac{d\psi}{dt} &= 50{\cdot}358\,25'' + x - 0{\cdot}000\,217\,612''\,t \\
\frac{d\psi_1}{dt} &= 50{\cdot}235\,72'' + x + 0{\cdot}000\,225\,800''\,t \\
\frac{d\lambda}{dt} &= 0{\cdot}133\,57'' - 0{\cdot}000\,483\,496''\,t \\
\sin\omega &= 0{\cdot}398\,088\,12 + 0{\cdot}000\,000\,000\,031\,9\,t^2 \\
\cos\omega &= 0{\cdot}917\,347\,17 - 0{\cdot}000\,000\,000\,013\,9\,t^2
\end{aligned}
\qquad (82)
$$

And by substituting these values in the well-known expressions

$$m = \frac{d\psi}{dt}\cos\omega - \frac{d\lambda}{dt} \qquad\qquad n = \frac{d\psi}{dt}\sin\omega$$

we obtain

$$
\begin{aligned}
m &= 46{\cdot}062\,43'' + 0{\cdot}000\,283\,870''\,t + 0{\cdot}917\,347\,x \\
n &= 20{\cdot}047\,02'' - 0{\cdot}000\,086\,629''\,t + 0{\cdot}398\,088\,x
\end{aligned}
\qquad (83)
$$

For the epoch 1777·5, from a comparison of his own reduction of BRADLEY'S observations with PIAZZI'S catalogue for 1800, BESSEL found*

$$m = 46{\cdot}034\,002'' \qquad\qquad n = 20{\cdot}064\,472$$

When NEWCOMB'S correction for systematic errors in the right ascensions of the catalogues is applied,† namely, $-0{\cdot}43''/45 = -0{\cdot}000\,555''$, these numbers become

$$m = 46{\cdot}024\,447'' \qquad\qquad n = 20{\cdot}064\,472$$

and their substitution in (83), together with $t = -72{\cdot}5$ years, gives

$$\text{from } m,\ x = -0{\cdot}018\,97'' \qquad\qquad \text{from } n,\ x = +0{\cdot}028\,06$$

Giving the result from m double weight, and taking the mean

$$x = -0{\cdot}003\,29'' \qquad (84)$$

For the epoch 1790, from a comparison of BESSEL'S reduction of BRADLEY'S observations with the Dorpat observations, OTTO STRUVE found the general precession to be $50{\cdot}234\,92'' \pm 0{\cdot}007\,57''$.‡ Applying NEWCOMB'S correction for systematic errors,§ namely, $-0{\cdot}37''/70 = -0{\cdot}005\,29''$, the precession becomes $50{\cdot}229\,63''$. Formula (82) gives for the same epoch $50{\cdot}222\,17''$, and therefore

$$x = +0{\cdot}007\,46'' \qquad (85)$$

For the epoch 1844·7, from a comparison of WEISSE'S reduction of BESSEL'S zones with SCHJELLERUP'S catalogue, NYRÉN found‖ $m = 46{\cdot}026\,5''$. Applying NEWCOMB'S

*76, p. 404. †81, p. 108. ‡85, p. 104. §81, p. 108. ‖82, p. 571.

correction for systematic errors, namely,* $+1.04''/36 = +0.028\,9''$, this becomes $m = 46.055\,4''$, which being substituted in (83), together with $t = -5.3$ years, gives

$$x = -0.006\,03'' \qquad (86)$$

For the epoch 1829·7, from a comparison of LALANDE'S zones with SCHJELLERUP'S catalogue, DREYER found† $m = 46.066\,6''$, which being substituted in (83), together with $t = -20.3$ years, gives

$$x = +0.010\,82'' \qquad (87)$$

For the epoch 1805·0 Mr. LUDWIG STRUVE made a comparison between Dr. AUWER'S reduction of BRADLEY'S observations and the PULKOWA catalogue of 1855, from which he found‡

$$m = 46.041\,7'' + 0.000\,274\,1''\,t$$

where t is reckoned from the year 1800. His observed result for 1805·0 must therefore have been

$$m = 46.043\,07''$$

and the substitution of that in (83), together with $t = -45.0$ years, gives

$$x = -0.007\,18'' \qquad (88)$$

Mr. L. STRUVE speaks highly of Dr. BOLTE'S Untersuchungen über die Präcessionsconstante,§ but the present writer has never seen that work.

Collecting our results, we have from numbers (84), (85), (86), (87), and (88)

Authority.	Value of x.	General Precession.	
		1800.	1850.
	//	//	//
BESSEL	—0·003 29	50·221 14	50·232 43
O. STRUVE	+0·007 46	50·231 89	50·243 18
NYRÉN	—0·006 03	50·218 40	50·229 69
DREYER	+0·010 82	50·235 25	50·246 54
L. STRUVE	—0·007 18	50·217 25	50·228 54
Means	+0·000 36	50·224 79	50·236 08

And by substituting the mean value of x in (77)

$$\begin{aligned} \psi\ &= (50.358\,61'' \pm 0.002\,48'')\,t - 0.000\,108\,806''\,t^2 \\ \psi_1 &= (50.236\,08'' \pm 0.002\,48'')\,t + 0.000\,112\,900''\,t^2 \end{aligned} \qquad (89)$$

*81, p. 109. †77, p. 154. ‡86, p. 30. § An inaugural dissertation, Bonn, 1883.

14.—THE CONSTANT OF NUTATION.

The following is an abstract of the most important determinations of the constant of nutation hitherto made:

	″ ″
1821. BRINKLEY, from 1618 zenith distances of 10 stars, measured at the observatory of Trinity College, Dublin, between the years 1808 and 1820. (90, p. 347.)	9·25 ± 0·05
1836. BUSCH, from 1949 zenith distances of 23 stars, observed at Kew and Wansted by BRADLEY during the years 1727 to 1747. (91, p. 338.)	9·232 ± 0·031
1838. ROBINSON, from 6023 zenith distances of 15 stars, measured at Greenwich with the mural circle during the years 1812 to 1835. (113, p. 18.)	9·239 ± 0·052
1841. LUNDAHL, from more than 1200 zenith distances of Polaris, observed at Dorpat during the years 1822 to 1838. (99½, p. 33.)	9·236 ± 0·040
1841. C. A. F. PETERS, from 603 right ascensions of Polaris, observed at Dorpat by STRUVE and PREUSS during the years 1822 to 1838. (109, p. 161.)	9·216 ± 0·020
1855. MAIN, from 173 zenith distances of γ Draconis, observed at Greenwich with the 25-foot zenith tube during the years 1837 to 1847. (102, p. 186.)	9·323 ± 0·059
1868. E. J. STONE, from 3250 zenith distances of Polaris, 51 Cephei, and δ Ursæ Minoris, together with 1936 right ascensions of Polaris, all observed at Greenwich, with the transit circle, during the years 1851 to 1867. (114, p. 249.)	9·134 ± 0·011
1871. NYRÉN, from 375 observations of υ Ursæ Majoris, ι Draconis, and o^2 Draconis, made at Pulkowa with the prime vertical transit instrument, during the years 1840 to 1862. (104, p. 30.)	9·244 ± 0·012
1882. DOWNING, from 1041 zenith distances of γ Draconis, observed at Greenwich with the reflex zenith tube, during the years 1857 to 1875. (92, p. 344.)	9·335 ± 0·032
1885. DE BALL, from 1867 right ascensions of Polaris, 51 Cephei, and δ Ursæ Minoris, observed at Pulkowa by WAGNER during the years 1861 to 1872. (89, p. 42.)	9·217 ± 0·012

The probable errors attached to BRINKLEY'S and ROBINSON'S results have been taken from PETERS'S paper,* and that attached to STONE'S result has been computed by the present writer.

Giving the determinations by MAIN and DOWNING half weight, because they rest upon a single star, we have from the weighted mean of the whole series

$$\mathfrak{N} = 9{\cdot}2331'' \pm 0{\cdot}0112'' \qquad (90)$$

BUSCH'S determination should probably have been rejected on account of the errors discovered in his computations by Dr. AUWERS,† but its retention does not sensibly affect (90). The probable error of (90) is largely increased by the constant errors which evidently exist in the results found by MAIN, STONE, and DOWNING; but as their determinations rest upon more than 6400 observations, made with three different instruments, it does not seem prudent to ignore them.

15.—THE CONSTANT OF ABERRATION.

The following is an abstract of the best values hitherto obtained for the constant of aberration:

	″ ″
1817. BESSEL, from his discussion of BRADLEY'S observations. (1, p. 123.)	20·475
1819. PIAZZI, by the observations made at Palermo (5, p. 207)	20·229
1821. BRINKLEY, from 2633 zenith distances of 14 stars observed at Trinity College, Dublin. (90, p. 350.)	20·372
1822. F. G. W. STRUVE, from 693 differences of right ascension between 6 pairs of stars observed at Dorpat. (115, p. lxiv.)	20·349

This result was subsequently corrected by C. A. F. PETERS to 20·361″ ± 0·0186″. See 111, p. 55.

* 109, p. 132. † 88, p. 611.

		″ ″
1828.	RICHARDSON, from 4119 zenith distances of 14 stars, observed at Greenwich. (112, p. 68.)	20·503
1836.	BUSCH, from 1949 zenith distances of 23 stars, observed by BRADLEY at Kew and Wansted. (91, p. 338.)	20·212 ± 0·038
1839.	HENDERSON, from 231 zenith distances of Sirius, observed at the Cape of Good Hope. (95, p. 248.)	20·41
1841.	F. G. W. STRUVE, from 19 observations of υ Ursæ Majoris in the prime vertical, at Pulkowa. (116, p. 290.)	20·493 ± 0·040
1841.	LINDENAU, from 800 right ascensions of Polaris, observed at Greenwich, Konigsberg, Dorpat, Palermo, Milan, and Seeberg, during the years 1750 to 1816. (98, p. 62, and 111, p. 65.)	20·449 ± 0·032
1841.	C. A. F. PETERS, from 603 right ascensions of Polaris, observed at Dorpat. (109, pp. 142 and 180.)	20·425 ± 0·017
1841.	LUNDAHL, from more than 1200 declinations of Polaris, observed at Dorpat. (99½, p. 37.)	20·551 ± 0.043
1842.	HENDERSON, from 272 double altitudes of α^1 and α^2 Centauri, observed at the Cape of Good Hope. (96, p. 370.)	20·523 ± 0·065
1843.	F. G. W. STRUVE, from 298 observations made upon 7 stars with the prime vertical transit instrument at Pulkowa. (118, p. 275.)	20·445 ± 0·011
1849.	C. A. F. PETERS, from BRADLEY'S sector observations at Greenwich. (111, p. 23.)	20·522 ± 0·079
1849.	C. A. F. PETERS, from 704 declinations of 8 stars observed with the Ertel vertical circle at Pulkowa. (111, p. 138.)	20·481 ± 0·013
1849.	LINDHAGEN, from 396 right ascensions of Polaris, observed at Pulkowa. (99, p. 354.)	20·498 ± 0·012
1851.	MACLEAR, from 391 double altitudes of α^1 and α^2 Centauri, observed at the Cape of Good Hope. (100, p. 98.)	20·531 ± 0·038
1852.	MACLEAR, from 137 double altitudes of β Centauri, observed at the Cape of Good Hope. (101, p. 152.)	20·594 ± 0·049
1860.	MAIN, from 486 zenith distances of γ Draconis, observed at Greenwich with the reflex zenith tube, during the years 1852–1859. (103, p. 190.)	20·335 ± 0·023
1882.	DOWNING, from 1041 zenith distances of γ Draconis, observed at Greenwich with the reflex zenith tube, during the years 1857–1875. (92, p. 344.)	20·378 ± 0·040
1883.	NYRÉN, from the series of observations made at Pulkowa, with the vertical circle, by PETERS, GYLDÉN, and NYRÉN; with the transit instrument, by SCHWEIZER and WAGNER; and with the prime vertical transit instrument, by F. G. W. STRUVE and NYRÉN. (106, p. 47.)	20·492 ± 0·006
1888.	A. HALL, from 436 observations of α Lyræ, made with the prime vertical transit instrument at Washington, during the years 1862–1867. (94, p. 12.)	20·454 ± 0·014
1888.	KÜSTNER, from 244 differences of meridian zenith distance of 7 pairs of stars, measured with the universal transit at Berlin. (97, p. 45.)	20·313 ± 0·011

A clearer exhibition of the facts will be attained by arranging the foregoing data somewhat differently, and in so doing PETERS's reductions of BRADLEY's observations at Greenwich and STRUVE's observations at Dorpat will be accepted to the exclusion of those by BESSEL and STRUVE himself; the determinations by PIAZZI and BRINKLEY will be omitted, the latter on account of the unexplained systematic errors exhibited by his declinations; BUSCH's reduction of BRADLEY's observations at Kew and Wansted will be omitted on account of the errors discovered in BUSCH's computations by AUWERS;* STRUVE's preliminary result from his observations of υ Ursæ Majoris in the prime vertical will also be omitted because the same observations are embodied in his final result, published in 1843; and NYRÉN's result will be separated into its original constituents. When these changes are effected, and the results are classed under the observatories furnishing the observations, we have the following exhibit:

GREENWICH.

	″
PETERS, from BRADLEY's sector observations	20·522
RICHARDSON, from mural circle observations	20·503
MAIN and DOWNING, from zenith distances of γ Draconis	20·356

CAPE OF GOOD HOPE.

HENDERSON, from Sirius	20·41
HENDERSON and MACLEAR, from α Centauri	20·527
MACLEAR, from β Centauri	20·594

*88, p. 611.

DORPAT.

	″
F. G. W. STRUVE, from transit observations	20·361
LUNDAHL, from declinations of Polaris	20·550
C. A. F. PETERS, from right ascensions of Polaris	20·425

PULKOWA.

F. G. W. STRUVE, from prime vertical observations	20·445
C. A. F. PETERS, from vertical circle observations	20·481
LINDHAGEN, from transit observations	20·498
NYRÉN, from GYLDÉN'S observations with the vertical circle	20·469
NYRÉN, from his own observations with the vertical circle	20·495
NYRÉN, from WAGNER'S observations with the transit	20·483
NYRÉN, from his own observations with the prime vertical transit	20·517

BERLIN.

KÜSTNER, from the zenith-telescope method	20·313

WASHINGTON.

A. HALL, from observations of α Lyræ with the prime vertical transit	20·454

VARIOUS OBSERVATORIES.

LINDENAU, from right ascensions of Polaris	20·449

Unmistakable evidences of constant errors are exhibited in these values of the constant of aberration—most notably in those found by MAIN, DOWNING, MACLEAR from β Centauri, F. G. W. STRUVE at Dorpat, and KÜSTNER. Nevertheless, it is difficult to assign thoroughly satisfactory reasons for rejecting any of them. The Pulkowa values are probably the most correct, but even they exhibit a range of 0·072″. Perhaps no two astronomers would assign the same relative weights to the different determinations, and yet, within rather wide limits, it is precisely these weights which determine the magnitude of the final result. If we take the means by observatories, we find

	″		″
Greenwich	20·460	Berlin	20·313
Cape of Good Hope	20·510	Washington	20·454
Dorpat	20·445	Miscellaneous	20·449
Pulkowa	20·484		

and the arithmetical mean of the results from all the observatories, except Berlin, is

$$\alpha = 20{\cdot}467'' \pm 0{\cdot}007''$$

Again, if we take the arithmetical mean of all the results in the last general exhibit, the Pulkowa values will have relatively a little more weight than the others, and we shall obtain

$$\alpha = 20{\cdot}466'' \pm 0{\cdot}011'' \qquad (91)$$

which will be adopted. This lies almost exactly midway between STRUVE'S classic value and that recently found by NYRÉN in his admirable paper on the aberration of the fixed stars.

16.—THE LIGHT EQUATION.

The time occupied by light in traversing the mean radius of the Earth's orbit is usually called the light equation, and there are but two determinations of it from the eclipses of Jupiter's satellites, namely, DELAMBRE'S, published in 1792, and GLASENAPP'S, published in 1874.

DELAMBRE'S value is $493^{\cdot}2^{s}$, which he originally derived from 500 eclipses of Jupiter's first satellite, and subsequently revised without obtaining any sensible correction, although he used more than a thousand eclipses of the same satellite. The details of these investigations have never been published, and our knowledge of them is confined to the brief allusions contained in LALANDE'S Astronomy,* DELAMBRE'S Astronomy,† and DELAMBRE'S Tables écliptiques des satellites de Jupiter.‡ The last is the most explicit.

GLASENAPP'S value is $500^{\cdot}84^{s} \pm 1^{\cdot}02^{s}$,§ which he derived from 391 eclipses of the first satellite of Jupiter, observed during the years 1848 to 1873. His memoir, although very valuable, is rendered almost inaccessible by being printed in the Russian language; but Mr. DOWNING has done something to remove that obstacle by publishing an excellent account of the work in The Observatory, vol. 12, pp. 173 and 210.

In combining these two results the following facts must not be overlooked:

1. The eclipses used by DELAMBRE were inferior to those used by GLASENAPP on account of having been observed with less powerful telescopes, and possibly with less accurate knowledge of local time; but their inferiority can not have been great, because DELAMBRE says of the eclipses he used, "il n'est pas rare de voir deux observations d'une même éclipse différer entre elles d'une demi minute,"‖ and residuals of that magnitude are not rare among GLASENAPP'S equations.

2. After a thorough trial of BAILLY'S photometric method of correcting observations of the eclipses of Jupiter's satellites, both MASKELYNE and DELAMBRE abandoned it as useless;¶ and yet GLASENAPP'S investigation is founded upon that very method, with some modifications whose value it is difficult to estimate. However, it will not escape notice that with the application of all his corrections GLASENAPP found the light equation to be $500^{\cdot}84^{s} \pm 1^{\cdot}02^{s}$, while without them he found it to be $497^{\cdot}15^{s} \pm 1^{\cdot}20^{s}$. The diminution effected in the probable error by the application of the corrections is so small as to indicate either the failure of BAILLY'S method or the existence of periodic errors in the tables of the motions of the satellites. GLASENAPP thought the latter hypothesis the more probable.

3. DELAMBRE'S result depends upon more than a thousand eclipses, while GLASENAPP'S depends upon only 391. And here we encounter the singular circumstance that the result which DELAMBRE obtained from 500 eclipses was not sensibly modified when he used more than a thousand; while GLASENAPP'S result, which rests upon nearly 400 eclipses, differs largely from DELAMBRE'S. The existence of constant errors in one or both series of observations seems the most probable explanation.

In view of these facts, it is not clear that GLASENAPP'S result is entitled to more confidence than DELAMBRE'S; nevertheless, we adopt the arithmetical mean of the two,

*6, T. 1, Tables astronomiques, p. 238.
†3, T. 3, pp. 105–106 and 502–507.
‡127, p. vij.
§129, p. 131, and 128, p. 211.
‖3, T. 3, p. 502.
¶3, T. 3, p. 507; 4, p. 746, and 12, vol. 1, p. 266, sec. 464.

thus giving GLASENAPP rather more than double weight, and in view of the uncertainty of the result we attribute the whole of GLASENAPP's probable error to it. In that way we find

$$\text{Light equation, } \theta = 497{\cdot}0^s \pm 1{\cdot}02^s \qquad (92)$$

which is almost identical with the result obtained by GLASENAPP when he omitted his corrections to the observed times of the eclipses.

17.—V, THE VELOCITY OF LIGHT IN VACUO.

The velocity of light can be measured between points upon the Earth's surface, either by the toothed-wheel method or by the revolving-mirror method. Both methods have been used, and the following are the principal results:

FIZEAU found

$$V = 70\,948 \text{ lieus of 25 to a degree} = 315\,324 \text{ kilometers} = 195\,935 \text{ miles}$$

His experiments were made in 1849 by the toothed-wheel method, working across an interval of 8·633 kilometers = 5·364 miles, between Suresnes and Montmartre, Paris.*

FOUCAULT's experiments gave†

$$V = 298\,574 \pm 204 \text{ kilometers} = 185\,527 \pm 127 \text{ miles}$$

He used the revolving-mirror method, and worked across an interval of only 20 meters = 65·6 feet, at Paris. His experiments were in progress from May to September, 1862; but he based his final result upon the 80 observations made on September 18, 19, and 21.

CORNU found, from the experiments which he made in August, 1872,

$$V = 298\,500 \pm 995 \text{ kilometers} = 185\,481 \pm 618 \text{ miles}$$

This result rests upon 658 experiments, made by the toothed-wheel method, working across an interval of 10·310 kilometers = 6·4064 miles, between l'École Polytechnique and Mont-Valérien, Paris.‡

CORNU found, from the experiments which he made in September, 1874,

$$V = 300\,400 \pm 300 \text{ kilometers} = 186\,662 \pm 186 \text{ miles}$$

This result rests upon 546 experiments, made by the toothed-wheel method, working across an interval of 22·910 kilometers = 14·2357 miles, between the observatory and Montlhéry, Paris.§

MICHELSON found, from his experiments of June and July, 1879,

$$V = 299\,910 \pm 51 \text{ kilometers} = 186\,357 \pm 31{\cdot}7 \text{ miles}$$

This result rests upon 100 experiments, made by the revolving-mirror method, working across an interval of 1986·23 feet = 0·3762 of a mile = 0·6054 of a kilometer, at the U. S. Naval Academy, Annapolis, Md.‖

*132, p. 92. †135, p. 224. ‡130, p. 178. §131, p. A. 298. ‖137, p. 157, and 139, p. 244.

YOUNG and FORBES found

$$V = 301\,384 \pm 263 \text{ kilometers} = 187\,273 \pm 164 \text{ miles}$$

This result rests upon only 12 experiments, made in December, 1880, and January, 1881, by a peculiar application of the toothed-wheel method. The light from the collimator containing the toothed wheel was sent simultaneously to two reflecting collimators situated at different distances, but nearly in the same straight line; and the observation consisted in determining when the images returned by these two collimators were of equal brightness. The toothed-wheel collimator was situated at Kelly House, Wemyss Bay, Scotland, and the reflecting collimators were located on the hills behind Innellan, across the mouth of the river Clyde, their distances being respectively 3·1884 and 3·4493 miles, or 5·1313 and 5·5510 kilometers.*

NEWCOMB found three results from the experiments which he made at Washington by the revolving-mirror method during the years 1880 to 1882. They are as follows:†

(*a*) From 148 experiments, made between June, 1880, and April, 1881, across an interval of 5·1019 kilometers = 3·1702 miles, between Fort Myer and the U. S. Naval Observatory,

$$V = 299\,709 \text{ kilometers} = 186\,232 \text{ miles}$$

(*b*) From 39 experiments, made in August and September, 1881, across an interval of 7·4424 kilometers = 4·6245 miles, between Fort Myer and the Washington Monument,

$$V = 299\,776 \text{ kilometers} = 186\,274 \text{ miles}$$

(*c*) From 65 experiments, made in July, August, and September, 1882, across the above-mentioned interval between Fort Myer and the Washington Monument,

$$V = 299\,860 \text{ kilometers} = 186\,326 \text{ miles}$$

If these results are to be combined, according to NEWCOMB we should assign the weight 2 to (*a*), 3 to (*b*), and 6 to (*c*); but he preferred to use (*c*) alone, on the ground that it is probably least affected by constant errors.

MICHELSON found, from his experiments in October and November, 1882,

$$V = 299\,853 \pm 60 \text{ kilometers} = 186\,322 \pm 37 \text{ miles}$$

This result rests upon 23 experiments, made by the revolving-mirror method, working across an interval of 2049·35 feet = 0·3881 of a mile = 0·6246 of a kilometer, at the Case Institute, Cleveland, Ohio.‡

Now let us examine CORNU's results a little more closely. His experiments cover a wide range in the speed of the toothed wheel, and his mean result for each speed is given in Table II, where V′ is the velocity of light derived from an experiment in which $\frac{1}{2}(2n - 1)$ teeth of the wheel passed during the interval between the departure and the return of the light, and p is the weight of V′. With respect to the experiments

*141, p. 269. †140, pp. 194, 201, and 202. ‡139, p. 244.

made in 1874, it is to be observed that the values of V′ are the weighted means of CORNU's uncorrected values of $\frac{1}{2}(V + v)$ and $\frac{1}{2}(U + u)$, the weights being taken just as CORNU gave them.

TABLE II.—*Cornu's experiments on the velocity of light.* (130, p. 174 and 131, pp. A. 266–7.)

Experiments of 1872.			Experiments of 1874.		
$2n - 1$	V′	p	$2n - 1$	V′	p
3	302·5	129	7	300 166	833
5	297·7	2 095	9	300 620	2 511
7	298·2	4 391	11	300 050	2 662
9	298·8	4 783	13	302 068	2 197
11	297·5	924	15	299 960	3 150
13	300·5	260	17	300 100	32 946
			19	300 224	28 880
			21	300 359	32 193
			23	300 500	1 587
			25	300 490	6 250
			27	300 304	13 122
			29	300 304	54 665
			31	299 874	6 727
			33	299 843	25 047
			35	300 083	29 400
			37	299 550	6 845
			41	300 097	45 387

HELMERT has pointed out that in the observations of 1874 V′ seems to diminish as $(2n - 1)$ increases; whence he infers that*

$$V' = V + \frac{y}{2n - 1} \tag{93}$$

where V is the true velocity of light, and y is a constant depending upon the conditions under which the experiments were made. If we put

$$V = C + x \qquad V' = C + m \tag{94}$$

and substitute these values in (93), the observation equations for determining V take the form

$$0 = x + \frac{y}{2n - 1} - m \tag{95}$$

and the weighted normal equations will be of the form

$$\begin{aligned} 0 &= [p]\,x + \left[\frac{p}{2n - 1}\right] y - [pm] \\ 0 &= \left[\frac{p}{2n - 1}\right] x + \left[\frac{p}{(2n - 1)^2}\right] y - \left[\frac{pm}{2n - 1}\right] \end{aligned} \tag{96}$$

* 136, p. 126.

Applying this to the observations made in 1872, with C = 298·4, the weighted normal equations are

$$0 = +\ 12\,582{\cdot}00x + 1\,724{\cdot}730y + 188{\cdot}200$$
$$0 = +\ \ \ 1\,724{\cdot}73x + \ \ 255{\cdot}969y + \ \ 63{\cdot}482$$

and their general solution is

$$x = -\,0{\cdot}001\,040\,912\ \mathrm{P} + 0{\cdot}007\,013\,710\ \mathrm{Q}$$
$$y = +\,0{\cdot}007\,013\,710\ \mathrm{P} - 0{\cdot}051\,165\,396\ \mathrm{Q}$$

where

$$\mathrm{P} = +\ 188{\cdot}200 \qquad\qquad \mathrm{Q} = +\ 63{\cdot}482$$

Whence

$$x = +\ 0{\cdot}249\,344\,700 \pm 0{\cdot}844\,7$$
$$y = -\ 1{\cdot}928\,101\,447 \pm 5{\cdot}922\,3 \qquad (97)$$
$$\mathrm{V} = 298\,649{\cdot}3 \pm 844{\cdot}7 \text{ kilometers}$$

with which the residuals in the normal equations are

$$+\ 0{\cdot}000\,606 \qquad\qquad +\ 0{\cdot}000\,085$$

the residuals in the weighted observation equations are

− 51·73	+ 11 53	+ 29·62
+ 25·82	− 25 26	− 32·19

and the probable error of an observation of weight unity is ± 26 183 kilometers.

Similarly, for the observations made in 1874, with C = 300 000, the weighted normal equations are

$$0 = +\ 294\,402x + 11\,808y - 51\,012\,357$$
$$0 = +\ \ \ 11\,808x + \ \ \ \ 546y - \ \ 2\,455\,681$$

and their general solution is

$$100x = -\,0{\cdot}002\,561\,621\,1\ \ \mathrm{P} + 0{\cdot}055\,398\,574\,2\ \ \mathrm{Q}$$
$$100y = +\,0{\cdot}055\,398\,574\,2 \qquad - 1{\cdot}381\,220\,446\,4$$

where

$$\mathrm{P} = -\ 51\,012\,357 \qquad\qquad \mathrm{Q} = -\ 2\,455\,681$$

Whence

$$x = -\ \ \ \ \ 53{\cdot}668\,960 \pm 111{\cdot}8 \text{ kilometers}$$
$$y = +\ 5\,658{\cdot}249\,626 \pm 2\,595 \text{ kilometers} \qquad (98)$$
$$\mathrm{V} = 299\,946{\cdot}3 \qquad \pm 111{\cdot}8 \text{ kilometers}$$

with which the residuals in the normal equations are + 5·4 and + 0·2; the residuals in the weighted observation equations are

+ 17 009	+ 3 420	+ 20 897
− 2 251	− 25 694	+ 43 500
+ 21 194	− 12 247	+ 4 287
− 79 091	− 25 102	+ 45 423
+ 20 395	− 16 975	− 2 698
+ 32 517	− 38 006	

and the probable error of an observation of weight unity is ± 22 080 kilometers.

The values of V in (97) and (98) require to be multiplied by 1·000 273 in order to reduce them to what they would have been in a vacuum; and thus we find

$$1872.\quad V = 298\,731 \pm 845 \text{ kilometers} = 185\,624 \pm 525 \text{ miles}$$
$$1874.\quad V = 300\,028 \pm 112 \text{ kilometers} = 186\,430 \pm 69{\cdot}5 \text{ miles}$$

which we shall employ instead of the numbers given by CORNU himself.

Collecting our results, we now have the following measurements of the velocity of light in vacuo per second of mean solar time, together with their estimated weights:

	Kilometers.	Miles.	Weight.
1849. FIZEAU	315 324	195 935	0
1862. FOUCAULT	298 574	185 527	1
1872. CORNU	298 731	185 624	1
1874. CORNU	300 028	186 430	2
1879. MICHELSON	299 910	186 357	3
1881. YOUNG and FORBES	301 384	187 273	1
1882. NEWCOMB	299 860	186 326	6
1882. MICHELSON	299 853	186 322	3

FIZEAU'S measurement is rejected on the ground that he himself regarded it as only a preliminary attempt. The weighted mean of all the other measurements gives

$$V = 299\,835 \pm 154 \text{ kilometers} = 186\,310 \pm 95{\cdot}6 \text{ miles} \qquad (99)$$

But if we consider only the four measurements whose weight is greater than unity, their weighted mean will give

$$V = 299\,893 \pm 23{\cdot}0 \text{ kilometers} = 186\,347 \pm 14{\cdot}3 \text{ miles} \qquad (100)$$

Thus it appears that in either case we arrive at substantially the same value of V, but with widely different probable errors. We shall adopt the value (100), with a probable error equal to the difference between (99) and (100), namely,

$$V = 299\,893 \pm 58 \text{ kilometers} = 186\,347 \pm 36 \text{ miles} \qquad (101)$$

18.—MASSES OF THE PLANETS.

We need the mass of the Earth as one of the elements for finding the solar parallax, and the masses of all the other planets are required in computing the luni-solar precession from the general precession. The following are some of the most noteworthy determinations of these masses; others may be found in HOUZEAU'S Vade-Mecum de l'Astronomie.

Reciprocals of the mass of Mercury.

1782. LA GRANGE, from his hypothetical relation between the densities of the planets and their distances from the Sun. (167, p. 190.)	2 025 810
1841. ENCKE, from the perturbations of the comet which bears his name, during its apparitions in 1819, 1825, 1828, 1835, and 1838, before perihelion. (153, p. 5.)	4 865 751
1851. ENCKE, from the perturbations of the comet which bears his name,	
(*a*) From the normal places in 1828 to 1848, excluding those of 1818, 1822, and 1825. (156, p. 47.)	10 252 900
(*b*) From the normal places before perihelion in 1828 to 1848, excluding those of 1818, 1822, 1825, 1832, and May, 1842. (156, p. 49.)	8 234 192
(*c*) From the normal places, 1818 to 1838, without distinction. (156, p. 51.)	3 200 448
(*d*) From all normal places without distinction, 1818 to 1848. (156, p. 51.)	3 271 742
ENCKE thought the mean of (*a*) and (*c*) must be near the truth, namely (156, p. 52)	4 878 172

1877.	Von Asten, from perturbations of Encke's Comet, 1818 to 1868. (145, p. 98.)	7 636 440
1886.	Backlund, from perturbations of Encke's Comet, 1871 to 1881. (146, p. 37.)	2 668 700
1842.	Rothman, from the motion of the perihelion of Venus. (184, p. 180.)	3 182 843
1861.	Le Verrier, from the perturbations of Venus. (171, p. 92.)	5 310 000
1861.	Le Verrier, in his tables of Mars. (8, T. 6, pp. 21 and 308.)	4 316 547
1881.	Tisserand, upon rediscussing certain of Le Verrier's equations, found that they were capable of two solutions. (189, p. 656.)	
	The one which he regarded as most probable gave	7 100 000
	The other gave	3 800 000
1882.	Newcomb, by estimation, from a consideration of all the data. (177, p. 468.)	7 500 000

Reciprocals of the mass of Venus.

1779.	La Grange,	
	From the motion of the Earth's perhelion. (166, p. 115.)	315 517
	From the periodic perturbations of the Earth. (166, p. 116.)	341 413
1802.	La Place, from the secular diminution of the obliquity of the ecliptic. (7, T. 3, liv. 6, chap. 6, § 21.)	383 137
1802.	Delambre, from the perturbations of the Earth. (7, T. 3, liv. 6, chap. 16.)	356 632
1803.	Wurm, from observations of the Sun. (191, p. 153.)	326 849
1813.	Burckhardt, from the pertuabations of the Earth. (151, p. 343.)	401 839
1813.	Lindenau, from the secular motions of the node and perihelion of Mercury. (173, p. 30.)	349 440
1827.	Airy, from observations of the Sun made at Greenwich. (142, p. 30.)	401 211
1842.	Rothman, from the secular motion of the node of Mercury. (184, p. 181.)	362 017
1843.	Le Verrier, from his equations for correcting the elements of Mercury. (169, p. 354.)	390 000
1853.	Hansen and Olufsen, in their tables of the Sun. (40, p. 1.)	408 134
1858.	Le Verrier, in his tables of the Sun. (41, p. 102.)	400 246
1861.	Le Verrier, in his tables of Mars. (8, T. 6, p. 309.)	412 150
1872.	Hill, from the motion of the node of Venus. (162, p. 36.)	427 240
1876.	Powalky, from observations of the Sun, made at Dorpat. (181, p. 265.)	396 980
1881.	Tisserand, from the variation of the obliquity of the ecliptic. (189, p. 658.)	425 500
1882.	Newcomb, by estimation, from a consideration of various results. (177, p. 472.)	405 000

Reciprocals of the mass of the Earth.

[Values of the Earth's mass obtained from the solar parallax do not come into consideration here.]

1832.	Plana, from the parallactic inequality of the Moon. (63, T. 3, p. 20.) (⊕ without ☾.)	352 359
1863.	Hansen, from the parallactic inequality of the Moon. (222, p. 11.) (⊕ without ☾.)	319 455
1872.	Le Verrier, from the action of the Earth on the other planets. (172, p. 169.) (⊕ + ☾.)	324 490
1881.	Tisserand, from a rediscussion of Le Verrier's equations. (189, p. 658.) (⊕ + ☾.)	325 700

Reciprocals of the mass of Mars.

1782.	La Grange, from his hypothetical relation between the densities of the planets and their distances from the Sun. (167, p. 190.)	1 846 082
1802.	Delambre, from the perturbations of the Earth. (7, T. 3, liv. 6, chap. 16.)	2 546 320
1813.	Burckhardt, from the perturbations of the Earth. (151, p. 343.)	2 680 337
1827.	Airy, from observations of the Sun made at Greenwich. (142, p. 30.)	3 734 602
1853.	Hansen and Olufsen, in their tables of the Sun. (40, p. 1.)	3 200 900
1858.	Le Verrier, in his tables of the Sun. (41, p. 102.)	2 994 790
1876.	Powalky, from observations of the Sun made at Dorpat. (181, p. 265.)	2 876 000
1876.	Le Verrier, in his tables of Jupiter. (8, T. 12, p. 9.)	2 812 526
1878.	A. Hall, from his own observations of the elongations of the satellites. (158, p. 37.)	3 093 500 ± 3 295

Reciprocals of the mass of Jupiter.

1726.	Newton, from the elongations of the fourth satellite, observed by Pound. (9, lib. 3, prop. 8, cor. 2.)	1 067
1782.	La Grange, from a recomputation of the same observations. (167, p. 183.)	1 067·195
1802.	La Place, from the same observations. (7, T. 3, liv. 6, chap. 6, § 21.)	1 067·09
1821.	Bouvard, from the perturbations of Saturn. (150, p. ij; compare also 8, T. 12, pp. 67–70.)	1 070·5
1823.	Nicolai, from the perturbations of Juno. (178, p. 226.)	1 053·924
1826.	Encke, from the perturbations of Vesta. (152, p. 267.)	1 050·36
1837.	Airy, from elongations of the fourth satellite, observed by himself. (143, p. 47.)	1 046·77
1842.	Bessel, from his heliometer measurements of the elongations of the fourth satellite. (147, p. 64.)	1 047·879 ± 0·158
1872.	Möller, from the perturbations of Faye's Comet. (175, p. 95.)	1 047·788 ± 0·185
1873.	Krueger, from the perturbations of Themis. (165, p. 14.)	1 047·538 ± 0·052
1881.	Schur, from his heliometer measurements of the elongations of all the four satellites. (186, p. 293.)	1 047·232 ± 0·246
1888.	Haerdtl, from the perturbations of Winnecke's Comet. (157, p. 262.)	1 047·175 ± 0·014

Reciprocals of the mass of Saturn.

1726. NEWTON, from the elongations of Titan. (9, lib. 3, prop. 8, cor. 2.)	3 021
1782. LA GRANGE, from the elongations of Titan. (167, p. 186.)	3 358·40
1802. LA PLACE, from the elongations of Titan. (7, T. 3, liv. 6, chap. 6, § 21.)	3 359·40
1821. BOUVARD, from the perturbations of Jupiter. (150, p. ij.)	3 512
1833. BESSEL, from his heliometer measurements of the elongations of Titan. (148, p. 24.)	3 501·6 ± 0·78
1876. LE VERRIER, in his tables of Jupiter. (8, T. 12, pp. 9 and 70–72.)	3 529·6
1885. A. HALL, from his observations of the elongations of Iapetus. (158½, p. 70.)	3 481·3 ± 0·54
1888. H. STRUVE, from his observations of the elongations of Iapetus and Titan. (187½, pp. 117–118.)	3 498·0 ± 1.17
1889. A. HALL, Jr., from his heliometer measurements of the elongations of Titan. (161, p. 146.)	3 500·5 ± 1·44

Reciprocals of the mass of Uranus.

1789. WURM, from HERSCHEL'S measurements of the elongations of the exterior satellite. (190, p. 214.)	16 959
1802. LA PLACE, from the same observations. (7, T. 3, liv. 6, chap. 6, § 21.)	19 504
1821. BOUVARD, in his Tables astronomiques. (150, p. ij.)	17 918
1838. LAMONT, from his measurements of the elongations of the second and fourth satellites. (168, p. 59.)	24 605
1871. VON ASTEN, from elongations of Oberon and Titania, observed by LAMONT, O. STRUVE, LASSELL, and MARTH. (144, p. 21.)	22 [illegible]
1875. Lord ROSSE and Dr. COPELAND, from their observations of Oberon and Titania. (182, p. 304.)	24 000
1875. NEWCOMB, from his observations of the elongations of all the four satellites. (176, p. 36.)	22 540 ± 50
1885. A. HALL, from his observations of the elongations of Oberon and Titania. (159, p. 33.)	22 682 ± 27

Reciprocals of the mass of Neptune.

1847. O. STRUVE, from his own observations of the satellite. (188, p. 815.)	14 494
1848. B. PEIRCE, from the perturbations of Uranus. (179, p. 205.)	20 000
1849. HIND, from elongations of the satellite, measured by BOND, LASSELL, and O. STRUVE. (163, p. 203.)	17 900
1850. G. P. BOND, from elongations of the satellite, observed at Cambridge, Mass. (149, p. 38.)	19 400
1854. HIND, from elongations of the satellite, observed by LASSELL in 1852. (164, p. 47.)	17 135
1862. SAFFORD, from the perturbations of Uranus. (185, p. 144.)	20 039 ± 295
1875. NEWCOMB, from his own observations of the elongations of the satellite. (176, p. 63.)	19 380 ± 70
1885. A. HALL, from the elongations of the satellite (160, p. 26):	
(*a*) From his own observations	19 092 ± 64
(*b*) From HOLDEN'S observations	18 279 ± 114
(*c*) From LASSELL and MARTH'S observations	17 850 ± 180

The various values given above for the masses of Mercury and Venus differ so largely among themselves that no trustworthy result can be deduced from them, and a rediscussion of the original data seems the only satisfactory course. For the masses of the planets outside the Earth the following numbers will be adopted:

$$\text{Mass of Mars} = \frac{1}{3\,093\,500} \qquad (102)$$

$$\text{Jupiter} = \frac{1}{1\,047{\cdot}55} \qquad (103)$$

$$\text{Saturn} = \frac{1}{3\,501{\cdot}6} \qquad (104)$$

$$\text{Uranus} = \frac{1}{22\,600} \qquad (105)$$

$$\text{Neptune} = \frac{1}{18\,780} \qquad (106)$$

Number (102) is Professor HALL'S value. For the mass of Jupiter the arithmetical mean of the values given by BESSEL, MÖLLER, KRUEGER, SCHUR, and HAERDTL is 1 : 1 047·522, while the mean of the two values given by BESSEL and SCHUR is 1 : 1 047·555. Number (103) is sensibly the latter value. According to H. STRUVE,*

* 187½, pp. 118–119.

the revised value of BESSEL's mass of Saturn is 1 : 3 502·5. The mean of that, and the value found by Mr. A. HALL, Jr., is 1 : 3 501·5, which differs so little from BESSEL's own value that the latter has been retained in (104). Number (105) is very nearly the arithmetical mean of the values given by NEWCOMB and HALL, and (106) is almost the arithmetical mean of the values given by BOND, NEWCOMB, and HALL, the latter including the observations by HOLDEN, LASSELL, and MARTH.

We now proceed to determine the masses of Mercury, Venus, and the Earth from the perturbations, both periodic and secular, of these planets and of Mars. To facilitate the discussion of the motions of the nodes of Mercury and Venus, let

Ω = longitude of the planet's ascending node,
Ω_0 = approximate value of Ω, such that $\Omega = \Omega_0 + \Delta\Omega$,
m = mass of the planet,
m_0 = approximate value of m, such that $m = m_0(1 + \nu)$,
i = inclination of the planet's orbit to the plane of the ecliptic,
t = time in Julian years of $365\frac{1}{4}$ days, counted from a specified epoch,
$d\lambda\odot$ = correction to the assumed value of the Sun's longitude,
$d\beta\odot$ = correction to the assumed value of the Sun's latitude,
di = correction to the assumed value of i,
δp and δq = certain coefficients whose numerical values are given by LE VERRIER in the Annales of the Paris Observatory, Tome 2, pp. 100–102.

Further, let symbols relating to the different planets be distinguished by superior Roman numerals, in the usual way, those relating to Mercury being without any numeral, while those which relate to Venus are marked $^{\mathrm{i}}$, those which relate to the Earth $^{\mathrm{ii}}$, and so on to Neptune, symbols relating to which will be distinguished by the numeral $^{\mathrm{vii}}$.

In order to make use of Professor NEWCOMB's investigation of the longitude of the node of Mercury, let us put

$$\Omega = \Omega_0 + \Delta\Omega \tag{107}$$

where the value of Ω_0 is that given by LE VERRIER for the epoch 1850·0, in his tables of Mercury,* namely,

$$\Omega_0 = 46^\circ\ 33'\ 08{\cdot}75'' + 42{\cdot}643''t + 0{\cdot}000\,083\,5''t^2 \tag{108}$$

From a discussion of 23 transits of Mercury, occurring between the years 1677 and 1881, NEWCOMB found for the epoch 1820·0†

$$N = N_0 + N't = -0{\cdot}16'' \pm 0{\cdot}18'' + (0{\cdot}28'' \pm 0{\cdot}42'')t$$

Whence, for the epoch 1850·0

$$N = -0{\cdot}16'' \pm 0{\cdot}18'' + (0{\cdot}28'' \pm 0{\cdot}42'')\frac{1850-1820}{100} + (0{\cdot}28'' \pm 0{\cdot}42'')t$$
$$= -0{\cdot}076'' \pm 0{\cdot}220'' + (0{\cdot}28'' \pm 0.42'')t$$

NEWCOMB wrote

$$N = (\delta\theta - \delta l')\sin i$$

* 170, p. 107. † 177, p. 460.

but, as $\delta l'$ can not be evalued, when the unit of time is changed from a century to a year this may be regarded as equivalent to

$$\varDelta\Omega = \frac{N_0}{\sin i} + \frac{N't}{100 \sin i}$$

which, with $i = 7^\circ\ 00'\ 07{\cdot}7''$, and the above numerical value of N, gives

$$\varDelta\Omega = -0{\cdot}623'' \pm 1{\cdot}805'' + (0{\cdot}023'' \pm 0{\cdot}034'')t \qquad (109)$$

By substituting (108) and (109) in (107), we obtain definitively for Mercury

$$\Omega = 46^\circ\ 33'\ 08{\cdot}13'' \pm 1{\cdot}80'' + (42{\cdot}666'' \pm 0{\cdot}034'')t + 0{\cdot}000\,083\,5''t^2 \qquad (110)$$

where t is counted in Julian years of $365\frac{1}{4}$ days, from 1850·0, Paris mean time.

Passing next to Venus, ENCKE found from the transit of 1761*

Epoch, 1761, June 5^d 17^h 30^m Paris mean time.

	° ′ ″
Longitude of Sun	75 35 49·6
Latitude of Sun	+0·6
Inclination of Venus's orbit . . .	3 23 26

$$\Omega^{\rm i} = 74^\circ\ 31'\ 54{\cdot}46'' + d\lambda\odot - 16{\cdot}882\,d\beta\odot - 10{\cdot}919\,d\rho + 0{\cdot}313\,di^{\rm i}$$

And from the transit of 1769*

Epoch, 1769, June 3^d 10^h 10^m Paris mean time.

	° ′ ″
Longitude of Sun	73 27 13·8
Latitude of Sun	0·0
Inclination of Venus's orbit	3 23 26

$$\Omega^{\rm i} = 74^\circ\ 36'\ 08{\cdot}60'' + d\lambda\odot - 16{\cdot}882\,d\beta\odot + 10{\cdot}451\,d\rho - 0{\cdot}339\,di^{\rm i}$$

In order to reduce these results to conformity with the positions of the Sun given by the Tables du Soleil of HANSEN and OLUFSEN, and to the inclination of the orbit of Venus given by HILL's tables of that planet, we take from the latter work, Introduction, pp. 2 and 23:

	Transit of 1761.	Transit of 1769.
	° ′ ″	° ′ ″
Longitude of Sun	75 35 52·05	73 27 14·25
Latitude of Sun	+0·53	−0·04
$i^{\rm i}$	3 23 31·64	3 23 31·94

* 202, pp. 105–107.

Whence, by comparison with ENCKE's results, and from HILL's tables of Venus,

	Transit of 1761.	Transit of 1769.
	″	″
$d\lambda\odot$	$+2.45$	$+0.45$
$d\beta\odot$	-0.07	$+0.04$
di^{i}	$+5.64$	$+5.94$
Perturbations of Venus in latitude	$+0.060$	$+0.084$
	° ′ ″	° ′ ″
ENCKE's observed ☊^{i}	74 31 54.46	74 36 08.60
Correction for $d\lambda\odot$	$+2.45$	$+0.45$
Correction for $d\beta\odot$	$+1.182$	-0.675
Correction for di^{i}	$+1.765$	-2.014
Correction for perturbations	$+1.013$	$+1.418$
Corrected ☊^{i}	74 32 00.87	74 36 07.78

The provisional expression for the longitude of the node of Venus, employed by HILL in the construction of his tables of that planet, was

$$☊^{i} = 75^{\circ}\ 19'\ 52.3'' + 32.2931''t + 0.000\ 151\, t^{2}$$

where t is reckoned in Julian years from the epoch 1850.0. To this, from the meridian observations made at Greenwich, Paris, and Washington during the years 1836 to 1871, he found the correction*

$$\varDelta ☊^{i} = \frac{w}{\sin\, i} = + \frac{0.12''}{0.059\ 18} = + 2.028''$$

which belongs to the epoch 1855, January 0.0, Washington mean time. Whence, for that epoch,

$$☊^{i} = 75^{\circ}\ 22'\ 35.78''$$

From the 1475 photographs of the last transit of Venus, reduced by the United States Transit of Venus Commission, we have for the epoch 1882, December $6^{d}\ 5^{h}\ 0^{m}$, Greenwich mean time,

$$☊^{i} = 75^{\circ}\ 37'\ 33.911'' + d\lambda\odot - 16.868\, d\beta\odot - 0.321\, di^{i}$$

This result already depends upon the position of the Sun given in the Tables du Soleil of HANSEN and OLUFSEN, and it requires correction only for the perturbations of Venus in latitude. According to HILL's tables of Venus, these perturbations amount to $+0.046''$, and the corrected result is, therefore,

$$☊^{i} = 75^{\circ}\ 37'\ 34.69''$$

* 162, Introduction, p. 36.

From HILL's tables of Venus, Introduction, p. 2, we have

$$☊^{i} = 75° \ 19' \ 53''\!\cdot\!10 + 32''\!\cdot\!5150 \, t + 0''\!\cdot\!000\,151 \, t^2$$

where

$$t = (\text{Date in Washington mean time}) - 1850\cdot0$$
$$= (\text{Date in Greenwich mean time}) - 1850\cdot000\,586$$

Computing the values of ☊ⁱ for the dates of observation by this expression, and collecting our observed results, we have

Julian date.	Computed ☊ⁱ	Observed ☊ⁱ	C — O
	° ′ ″	° ′ ″	″
1761·398 967	74 31 54·49	74 32 00·87	— 6·38
1769·392 654	74 36 14·20	74 36 07·78	+ 6·42
1854·966 363	75 22 35·66	75 22 35·78	— 0·12
1882·897 900	75 37 44·01	75 37 34·69	+ 9·32

In order to form the observation equations for the determination of the corrected expression for ☊ⁱ, let

$$0 = 75° \ 19' \ 53''\!\cdot\!10 + x + (32''\!\cdot\!5150 + y)t + 0\cdot000\,151\,t^2 - (\text{observed ☊}^{i} \text{ at the time } t) \qquad (111)$$

and then, by putting C for the computed, and O for the corresponding observed value of ☊ⁱ, we shall have

$$0 = (\mathrm{C} - \mathrm{O}) + x + ty$$

The scale of the Julian dates given above is such that 1849·967 042 corresponds to 1850·0 Washington mean time. Bearing that in mind, the following observation equations result from the values of (C — O):

$$0 = -6''\!\cdot\!38 + x - 88\cdot568y \qquad \text{Weight } 1$$
$$0 = +6\cdot42 + x - 80\cdot574y \qquad \text{Weight } 1$$
$$0 = -0\cdot12 + x + \ 4\cdot999y \qquad \text{Weight } 1$$
$$0 = +9\cdot32 + x + 32\cdot931y \qquad \text{Weight } 4$$

The probable error of the observed right ascension of Venus was ±0·470″ in 1761, ±0·496″ in 1769, and only ±0·038″ in 1882. Therefore, according to theory the weight of the last observation equation would be more than 150 times that of either of the first two. Nevertheless, on account of the possible existence of constant errors, it has been thought prudent to give it a weight of only four.

Having regard to the adopted weights, the normal equations are

$$\begin{aligned} 0 &= +\ \ 37{\cdot}200'' + 7{\cdot}000x - 32{\cdot}419y \\ 0 &= +\ 1274{\cdot}847'' - 32{\cdot}419x + 18\,699{\cdot}253y \end{aligned} \qquad (112)$$

and the general solution is

$$\begin{aligned} x &= -\ 0{\cdot}144\,013\,468\,\mathrm{P} - 0{\cdot}000\,249\,677\,\mathrm{Q} \\ y &= -\ 0{\cdot}000\,249\,677\,\mathrm{P} - 0{\cdot}000\,053\,911\,\mathrm{Q} \end{aligned}$$

in which

$$\mathrm{P} = +\ 37{\cdot}200'' \qquad \mathrm{Q} = +\ 1274{\cdot}847''$$

From the general solution

$$x = -\ 5{\cdot}675\,601'' \pm 1{\cdot}973'' \qquad y = -\ 0{\cdot}078\,016\,2'' \pm 0{\cdot}038\,2''$$

with which values the residuals in the normal equations are respectively 0·0000″ and — 0·0004″, while those in the weighted observation equations are respectively — 5·146,″ + 7·030″, — 6·186″, and + 2·151″.

If it is desired to use only the data afforded by transits of Venus, we may take $\frac{1}{2}(1761 + 1769)$ and 1882. In that way $d\rho$, the unknown correction to the adopted value of the Sun's semi-diameter, is sensibly eliminated from the observations of 1761 and 1769, and our observation equations become

$$\begin{aligned} 0 &= +\ 0{\cdot}02'' + x - 84{\cdot}571\,y \\ 0 &= +\ 9{\cdot}32'' + x + 32{\cdot}931\,y \end{aligned}$$

Whence

$$x = -\ 6{\cdot}714'' \qquad y = -\ 0{\cdot}079\,147''$$

The difference between this result and that from the normal equations (112) is less than the probable error of either.

Reverting to the values of x and y yielded by the equations (112), and substituting them in (111), we obtain definitively for Venus

$$\Omega^1 = 75^\circ\ 19'\ 47{\cdot}42'' \pm 1{\cdot}97'' + (32{\cdot}4370'' \pm 0{\cdot}0382'')t + 0{\cdot}000\,151''t^2 \qquad (113)$$

which belongs to the epoch 1850·0 Washington mean time.

If we put with LE VERRIER*

$$p = \tan i \sin \Omega \qquad q = \tan i \cos \Omega$$

then, regarding all the quantities as variable, and differentiating

$$dp = \frac{\sin \Omega}{\cos^2 i}\, di + \tan i \cos \Omega{\cdot}d\Omega$$

$$dq = \frac{\cos \Omega}{\cos^2 i}\, di - \tan i \sin \Omega{\cdot}d\Omega$$

*8, T. 2, p. 26.

whence, by eliminating di, we find for the theoretical motion of the node on the fixed ecliptic

$$d\Omega = \cot i \cos \Omega \cdot dp - \cot i \sin \Omega \cdot dq$$

To find the theoretical motion on the movable ecliptic, it is only necessary to substitute $dp - dp^{ii}$ and $dq - dq^{ii}$ for dp and dq. In that way we obtain for Mercury and Venus

$$d\Omega = \cot i \cos \Omega \, (dp - dp^{ii}) - \cot i \sin \Omega \, (dq - dq^{ii}) \tag{114}$$

$$d\Omega^{i} = \cot i^{i} \cos \Omega^{i} \, (dp^{i} - dp^{ii}) - \cot i^{i} \sin \Omega^{i} \, (dq^{i} - dq^{ii}) \tag{115}$$

From equations (110) and (113), and from Le Verrier's tables of Mercury and Hill's tables of Venus, we have for the epoch 1850·0

$$\Omega = 46^\circ \; 33' \; 08{\cdot}13'' \qquad i = 7^\circ \; 00' \; 07{\cdot}71''$$
$$\Omega^{i} = 75^\circ \; 19' \; 47{\cdot}42'' \qquad i^{i} = 3^\circ \; 23' \; 35{\cdot}01''$$

By the substitution of these values, and after replacing dp, dp^{i}, dp^{ii}, dq, dq^{i}, and dq^{ii} by the numerical values of δp, δp^{i}, δp^{ii}, δq, δq^{i}, and δq^{ii}, from the Annales of the Paris Observatory, T. 2, pp. 100–101, equations (114) and (115) become

$$\begin{aligned} \frac{d\Omega}{dt} = &- 7{\cdot}606\,89'' - 0{\cdot}065\,90''\nu - 4{\cdot}102\,14''\nu^{i} - 0{\cdot}923\,46''\nu^{ii} \\ &- 0{\cdot}112\,38''\nu^{iii} - 2{\cdot}283\,53''\nu^{iv} - 0{\cdot}117\,04''\nu^{v} \\ &- 0{\cdot}001\,84''\nu^{vi} \\ \frac{d\Omega^{i}}{dt} = &- 17{\cdot}367\,93'' + 0{\cdot}111\,93''\nu - 5{\cdot}038\,55''\nu^{i} - 6{\cdot}717\,49''\nu^{ii} \\ &- 0{\cdot}221\,72''\nu^{iii} - 5{\cdot}222\,75''\nu^{iv} - 0{\cdot}273\,55''\nu^{v} \\ &- 0{\cdot}004\,18''\nu^{vi} \end{aligned} \tag{116}$$

Putting ψ_1 for the general precession, the observation equations for determining the masses of Mercury, Venus, and the Earth will be of the form

$$0 = \text{theoretical} \frac{d\Omega}{dt} + \psi_1 - \text{observed} \frac{d\Omega}{dt} \tag{117}$$

As the masses of all the planets outside the Earth's orbit are here regarded as known, the numerical values of their νs must be substituted in the observation equations. To find them, we have the relation

$$m = m_0(1 + \nu)$$

where m_0 is the value employed in the observation equations for the mass of any planet, or, in other words, it is the value employed by Le Verrier in forming the quantities δp, δq, etc., and m is the adopted mass of the same planet. The numerical values of $1 \div m_0$, $1 \div m$ and ν are given for each planet in Table III.

TABLE III.

Planet.	$1 \div m_0$ Le Verrier.	$1 \div m$ Adopted.	Factor ν
Mercury	3 000 000	Indeterminate	
Venus	401 847	Indeterminate	
Earth	354 936	Indeterminate	
Mars	2 680 337	3 093 500	— 0·133 558
Jupiter	1 050	1 047·55	+ 0·002 339
Saturn	3 512	3 501·6	+ 0·002 970
Uranus	24 000	22 600	+ 0·061 947
Neptune	14 400	18 780	— 0·233 227

Taking the observed values of the motions of the nodes from equations (110) and (113), the theoretical values from the equations (116), and the factors for the corrections of the masses from Table III, the formation of the absolute terms of the observation equations will be as follows:

	Mercury.	Venus.
	″	″
Provisional motion of node	— 7·606 89	— 17·367 93
Correction for mass of Mars	+ 0·015 01	+ 0·029 61
Correction for mass of Jupiter	— 0·005 34	— 0·012 21
Correction for mass of Saturn	— 0·000 35	— 0·000 81
General precession	+ 50·237 19	+ 50·237 19
Theoretical $d\Omega/dt$	+ 42·639 62	+ 32·885 85
Observed $d\Omega/dt$	+ 42·666 0	+ 32·437 0
(C — O) = absolute term	— 0·026 4	+ 0·448 9

The observation equations are therefore

$$\begin{aligned} 0 &= -0''.0659\,\nu - 4''.1021\,\nu^{i} - 0''.9235\,\nu^{ii} - 0''.0264 \pm 0''.034 \\ 0 &= +0.1119\,\nu - 5.0386\,\nu^{i} - 6.7175\,\nu^{ii} + 0.4489 \pm 0.038 \end{aligned} \qquad (118)$$

whence, leaving ν^{ii} indeterminate,

$$\begin{aligned} \nu &= +28.951\,626\,\nu^{ii} - 2.495\,943 \pm 0.427 \\ \nu^{i} &= -\ \ 0.690\,234\,\nu^{ii} + 0.033\,661 \pm 0.005\,75 \end{aligned}$$

If we assume $\nu^{ii} = +0.066\,631$, then

$$\begin{aligned} \nu &= -0.566\,867 \pm 0.427 \\ \nu^{i} &= -0.012\,330 \pm 0.005\,75 \end{aligned}$$

In the Annales of the Paris Observatory, T. 6, pp 286 and 307, Le Verrier has given four equations which may be used for determining the masses of Mercury, Venus, and the Earth. They are derived respectively from the secular motions of the perihelion, eccentricity, inclination, and node of Mars; and, after multiplication by 100, they are as follows:

$$
\begin{aligned}
0 &= +0.14''\nu + 4.66''\nu^{i} + 16.36''\nu^{ii} + 130.6''\nu^{iv} - 2.353'' \\
0 &= +0.08\nu + 0.69\nu^{i} + 2.06\nu^{ii} + 18.2\nu^{iv} - 1.115 \\
0 &= -0.10\nu + 12.22\nu^{i} + 0.03\nu^{ii} - 13.11\nu^{iv} + 0.565 \\
0 &= -0.69\nu - 25.60\nu^{i} - 6.82\nu^{ii} - 37.15\nu^{iv} - 0.577
\end{aligned} \qquad (119)
$$

Regarding ν, ν^{i}, and ν^{ii} as unknown, the normal equations are

$$
\begin{aligned}
0 &= + 0.5121''\nu + 17.1496''\nu^{i} + 7.1580''\nu^{ii} + 46.6845''\nu^{iv} - 0.0770'' \\
0 &= +17.1496\nu + 826.8801\nu^{i} + 252.6176\nu^{ii} + 1411.9898\nu^{iv} + 9.9411 \\
0 &= + 7.1580\nu + 252.6176\nu^{i} + 318.4065\nu^{ii} + 2427.0777\nu^{iv} - 36.8399
\end{aligned} \qquad (120)
$$

and the general solution is

$$
\begin{aligned}
\nu &= -7.084\,439\,P + 0.129\,718\,Q + 0.056\,347\,7\,R \\
10\nu^{i} &= +1.297\,176\,P - 0.039\,714\,Q + 0.002\,347\,2\,R \\
100\nu^{ii} &= +5.634\,770\,P + 0.023\,472\,Q - 0.459\,359\,4\,R
\end{aligned}
$$

in which

$$
\begin{aligned}
P &= + 46.6845\nu^{iv} - 0.0770 \\
Q &= + 1411.9898\nu^{iv} + 9.9411 \\
R &= + 2427.0777\nu^{iv} - 36.8399
\end{aligned}
$$

By substituting the value of ν^{iv} from Table III these quantities become

$$
P = +0.0321 \qquad Q = +13.2423 \qquad R = -31.1654
$$

and then, from the general solution,

$$
\begin{aligned}
\nu &= -0.265\,744 \pm 1.511 \\
\nu^{i} &= -0.055\,742 \pm 0.035\,76 \\
\nu^{ii} &= +0.148\,078 \pm 0.038\,47
\end{aligned}
$$

With these values of the νs, the residuals in the normal equations (120) are

$$
+0.000\,001'' \qquad +0.000\,055'' \qquad -0.000\,008''
$$

the residuals in the observation equations (119) are

$$
+0.0779'' \qquad -0.8272'' \qquad -0.1159'' \qquad -0.0634''
$$

and the probable error of any one of the observation equations (119) is $\pm 0.5675''$.

We have next to deal with the following group of equations:

$$
\begin{aligned}
0 &= 0.00''\nu + 29.5''\nu^{i} + 225.3''\nu^{ii} - 18.59'' \\
0 &= -27.39\nu - 46.33\nu^{i} - 51.59\nu^{ii} - 18.02 \\
0 &= +14.3\nu + 25.5\nu^{i} + 27.7\nu^{ii} + 1.7 \\
0 &= + 7.8\nu + 9.2\nu^{i} + 15.3\nu^{ii} + 3.7 \\
0 &= - 0.53\nu + 24.6\nu^{i} + 32.8\nu^{ii} - 1.86 \\
0 &= - 1.24\nu + 40.4\nu^{i} + 54.0\nu^{ii} - 3.28 \\
0 &= + 0.53\nu + 28.88\nu^{i} \quad 0.0\nu^{ii} + 1.74
\end{aligned} \qquad (121)
$$

When Mars made its near approach to ψ^2 Aquarii on October 1, 1672, the position of the planet was compared with that of the star by RICHER at Cayenne, by PICARD near Beaufort, and by ROEMER at Paris. From a very careful discussion of these comparisons, LE VERRIER derived the equation which he has given in the Comptes Rendus, T. 75, p. 169, namely,

$$0 = + 29.5'' \nu^{i} + 225.3'' \nu^{ii} + 1398'' \nu^{iv} - 21.86''$$

By substituting in it the value of ν^{iv} from Table III, the first equation of the group (121) results.

The second equation of the group (121) is from the Annales of the Paris Observatory, T. 6, p. 73; and is based upon the longitude of Venus deduced from HORROX's observation of the transit of that planet in December, 1639. The third and fourth equations are from the Annales, T. 6, p. 76; the former being derived from the longitudes of Venus obtained from BRADLEY's meridian observations, and the latter from the longitudes obtained from the meridian observations made between the years 1766 and 1830. The fifth and sixth equations are from the Annales, T. 6, p. 90; the former being derived from the latitudes of Venus resulting from the observations of the transits in 1761 and 1769, and the latter from the latitudes resulting from the Greenwich meridian observations made between the years 1751 and 1830.

In the Annales of the Paris Observatory, T. 4, p. 52, LE VERRIER has given the equation

$$0 = + 0.53'' \nu + 28.88'' \nu^{i} + 0.83'' \nu^{iii} + 1.81'' \qquad (122)$$

which he has derived from the secular diminution of the obliquity of the ecliptic. By adding the term $+ 16.01'' \nu^{iv}$, from p. 51 of the Annales, and substituting the values of ν^{iii} and ν^{iv} from Table III, the last equation of the group (121) results. In the Annales, T. 5, p. 100, and T. 6, p. 91, and in the Comptes Rendus, T. 75, p. 168, LE VERRIER has given equation (122) in the form

$$0 = + 0.53'' \nu + 28.88'' \nu^{i} + 0.75'' \nu^{iii} + 1.72'' \qquad (123)$$

but the difference arises solely from the circumstance that in (122) the assumed mass of Mars is 1 : 2 680 337, while in (123) it is 1 : 2 994 790. For further comments on the equations (121), TISSERAND's paper in the Comptes Rendus may be consulted.*

From the group of observation equations (121) the following normals result:

$$\begin{aligned} 0 &= + 1\,017.64'' \nu + 1\,657.55'' \nu^{i} + 1\,844.16'' \nu^{ii} + 552.713'' \\ 0 &= + 1\,657.55 \nu + 6\,822.98 \nu^{i} + 12\,872.10 \nu^{ii} + 235.835 \\ 0 &= + 1\,844.16 \nu + 12\,872.10 \nu^{i} + 58\,414.84 \nu^{ii} - 3\,393.103 \end{aligned} \qquad (124)$$

* 189.

and their general solution is

$$1\,000\,\nu = -1{\cdot}764\,489\,702\,P + 0\,553\,787\,566\,Q - 0\,066\,325\,742\,R$$
$$1\,000\,\nu^{i} = +0{\cdot}553\,787\,566\,P - 0{\cdot}424\,651\,923\,Q + 0{\cdot}076\,091\,780\,R$$
$$1\,000\,\nu^{ii} = -0{\cdot}066\,325\,742\,P + 0{\cdot}076\,091\,780\,Q - 0{\cdot}031\,792\,362\,R$$

where

$$P = +552{\cdot}713 \qquad Q = +235{\cdot}835 \qquad R = -3\,393{\cdot}103$$

We therefore have

$$\nu = -0\,619\,603\,8 \pm 0{\cdot}096\,92$$
$$\nu^{i} = -0{\cdot}052\,249\,4 \pm 0{\cdot}047\,55$$
$$\nu^{ii} = +0{\cdot}089\,160\,8 \pm 0{\cdot}013\,01$$

with which the residuals in the normal equations (124) are

$$+0{\cdot}000\,2'' \qquad +0{\cdot}000\,8'' \qquad +0{\cdot}002\,8''$$

The residuals in tne observation equations (121) are

$$-0{\cdot}043'' \qquad -6{\cdot}022'' \qquad +0{\cdot}107'' \qquad -0{\cdot}097''$$
$$-3{\cdot}228'' \qquad -0{\cdot}250'' \qquad +0{\cdot}192''$$

and the probable error of any one of the observation equations (121) is $\pm 2{\cdot}307''$.

The equations in group (125) are from the Annales of the Paris Observatory, T. 4, p. 95, and depend entirely upon observations of the Sun. The first equation of the group has been deduced from the differences of the maximum values of the equation of the center determined at two epochs fifty years apart. The second equation arises from the observed motion of the Earth's perigee; and the remaining equations are based upon the periodic perturbations of the Earth, the third and fifth arising from the action of Venus, while the fourth and sixth arise from the action of Mars.

$$\begin{aligned}
0 &= -0{\cdot}23''\,\nu + 1{\cdot}01''\,\nu^{i} - 0{\cdot}65''\,\nu^{iii} - 0{\cdot}21''\\
0 &= -0\,43\,\nu + 5{\cdot}97\,\nu^{i} + 1{\cdot}93\,\nu^{iii} - 0{\cdot}44\\
0 &= -0{\cdot}04\,\nu + 8{\cdot}00\,\nu^{i} - 0{\cdot}13\,\nu^{iii} - 0{\cdot}01\\
0 &= +0{\cdot}02\,\nu + 1{\cdot}07\,\nu^{i} + 4{\cdot}00\,\nu^{iii} + 0{\cdot}48\\
0 &= -0{\cdot}02\,\nu + 8{\cdot}00\,\nu^{i} - 0{\cdot}17\,\nu^{iii} - 0{\cdot}09\\
0 &= 0{\cdot}00\,\nu + 0{\cdot}61\,\nu^{i} + 4{\cdot}00\,\nu^{iii} + 0{\cdot}35
\end{aligned} \tag{125}$$

In accordance with LE VERRIER's estimate of the relative accuracy of these equations,* the second will be given a weight of $\frac{1}{9}$. It then becomes

$$0 = -0{\cdot}14''\,\nu + 1{\cdot}99''\,\nu^{i} + 0{\cdot}64''\,\nu^{iii} - 0{\cdot}15'' \tag{126}$$

*41, pp. 95–96.

and, regarding ν^{iii} as known, the weighted normal equations are

$$\begin{aligned} 0 &= +0\cdot0749''\nu - 0\cdot9695''\nu^{i} + 0\cdot1485''\nu^{iii} + 0\cdot0811'' \\ 0 &= -0\cdot9695''\nu + 134\cdot4972''\nu^{i} + 4\cdot9371''\nu^{iii} - 0\cdot5835'' \end{aligned} \qquad (127)$$

The general solution is

$$\begin{aligned} \nu &= -14\cdot725\,041\,P - 0\cdot106\,143\,Q \\ \nu^{i} &= -\;\;0\cdot106\,143\,P - 0\cdot008\,200\,Q \end{aligned}$$

where

$$\begin{aligned} P &= +0\cdot1485\,\nu^{iii} + 0\cdot0811 \\ Q &= +4\cdot9371\,\nu^{iii} - 0\cdot5835 \end{aligned}$$

Hence

$$\begin{aligned} \nu &= -2\cdot710\,707\,\nu^{iii} - 1\cdot132\,266 \\ \nu^{i} &= -0\cdot056\,246\,\nu^{iii} - 0\cdot003\,824 \end{aligned} \qquad (128)$$

and the residuals in the normal equations (127) are

$$\begin{aligned} &-0\cdot000\,00''\,\nu^{iii} + 0\cdot000\,00'' \\ &+0\cdot000\,20''\,\nu^{iii} - 0\cdot000\,09'' \end{aligned}$$

The substitution in (128) of the value of ν^{iii} from Table III gives

$$\begin{aligned} \nu &= -0\cdot770\,229 \pm 0\cdot317 \\ \nu^{i} &= +0\cdot003\,688 \pm 0\cdot007\,49 \end{aligned}$$

with which the residuals in the weighted observation equations (125) and (126) are

$$+0\cdot058'' \qquad -0\cdot120'' \qquad +0\cdot068'' \qquad -0\cdot065'' \qquad -0\cdot022'' \qquad -0\cdot182''$$

and the probable error of any one of these equations is $\pm 0\cdot082\,76''$.

Collecting our results, from the groups of equations (118), (119), (121), and (125), the values of ν are

$$\begin{aligned} &-0\cdot566\,867 \pm 0\cdot427 \\ &-0\cdot265\,744 \pm 1\cdot511 \\ &-0\cdot619\,604 \pm 0\cdot096\,92 \\ &-0\cdot770\,229 \pm 0\cdot317 \end{aligned}$$

and the values of ν^{i} are

$$\begin{aligned} &-0\cdot012\,330 \pm 0\cdot005\,75 \\ &-0\cdot055\,742 \pm 0\cdot035\,76 \\ &-0\cdot052\,249 \pm 0\cdot047\,55 \\ &+0\cdot003\,688 \pm 0\cdot007\,49 \end{aligned}$$

while from the groups of equations (119) and (121) the values of ν^{ii} are

$$\begin{aligned} &+0\cdot148\,078 \pm 0\cdot038\,47 \\ &+0\cdot089\,161 \pm 0\cdot013\,01 \end{aligned}$$

Instead of attempting to deduce final values from these results, it will be better to reduce the various groups of equations to a uniform standard of weight, and then to solve them all simultaneously. The data for that purpose are as follows:

Equations.	Probable error.	$\sqrt{\text{Weight}}$
Group (118) . .	± 0·036	10·000
Group (119) . .	± 0·567	0·635
Group (121) . .	± 2·307	0·156
Group (125) . .	± 0·0828	4·35

The weight of the equations in group (118) is arbitrarily assumed to be 100, and then the weights for the other groups follow from the probable errors given in the second column. By the application of these weights, each to its own group of equations, the subjoined system of weighted observation equations is obtained:

$$
\begin{array}{lllll}
 & '' & '' & '' & '' \\
0 = & -0·659\,\nu & -41·021\,\nu^{i} & -\ \ 9·235\,\nu^{ii} & -0·264 \\
0 = & +1·119\,\nu & -50·386\,\nu^{i} & -67·175\,\nu^{ii} & +4·489 \\
0 = & +0·089\,\nu & +\ \ 2·959\,\nu^{i} & +10·389\,\nu^{ii} & -1·300 \\
0 = & +0·051\,\nu & +\ \ 0·438\,\nu^{i} & +\ \ 1·308\,\nu^{ii} & -0·681 \\
0 = & -0·064\,\nu & +\ \ 7·760\,\nu^{i} & +\ \ 0·019\,\nu^{ii} & +0·339 \\
0 = & -0·438\,\nu & -16·256\,\nu^{i} & -\ \ 4·331\,\nu^{ii} & -0·422 \\
0 = & \ \ \ 0·000\,\nu & +\ \ 4·602\,\nu^{i} & +35·147\,\nu^{ii} & -2·900 \\
0 = & -4·273\,\nu & -\ \ 7·227\,\nu^{i} & -\ \ 8·048\,\nu^{ii} & -2·811 \\
0 = & +2·231\,\nu & +\ \ 3·978\,\nu^{i} & +\ \ 4·321\,\nu^{ii} & +0·265 \\
0 = & +1·217\,\nu & +\ \ 1·435\,\nu^{i} & +\ \ 2·387\,\nu^{ii} & +0·577 \\
0 = & -0·083\,\nu & +\ \ 3·838\,\nu^{i} & +\ \ 5·117\,\nu^{ii} & -0·290 \\
0 = & -0·193\,\nu & +\ \ 6·302\,\nu^{i} & +\ \ 8·424\,\nu^{ii} & -0·512 \\
0 = & +0·083\,\nu & +\ \ 4·505\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & +0·271 \\
0 = & -1·000\,\nu & +\ \ 4·394\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & -0·535 \\
0 = & -0·609\,\nu & +\ \ 8·656\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & -1·022 \\
0 = & -0·174\,\nu & +34·800\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & +0·030 \\
0 = & +0·087\,\nu & +\ \ 4·654\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & -0·235 \\
0 = & -0·087\,\nu & +34·800\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & -0·291 \\
0 = & \ \ \ 0·000\,\nu & +\ \ 2·654\,\nu^{i} & \ \ \ \ 0·000\,\nu^{ii} & -0·800
\end{array}
\qquad (129)
$$

The resulting normal equations are

$$
\begin{array}{lllll}
 & '' & '' & '' & '' \\
0 = & +28·0773\,\nu & -\ \ \ \ 0·4420\,\nu^{i} & -\ \ \ \ 21·3120\,\nu^{ii} & +\ \ 19·8172 \\
0 = & -\ \ 0·4420\,\nu & +7265·9429\,\nu^{i} & +4178·6252\,\nu^{ii} & -227·7718 \\
0 = & -21·3120\,\nu & +4178·6252\,\nu^{i} & +6147·7642\,\nu^{ii} & -394·2509
\end{array}
\qquad (130)
$$

and their general solution is

$$
\begin{aligned}
100\,\nu &= -3·576\,684\,283\,P + 0·011\,349\,517\,Q - 0·020\,113\,276\,R \\
100\,\nu^{i} &= +0·011\,349\,517\,P - 0·022\,631\,115\,Q + 0·015\,421\,676\,R \\
100\,\nu^{ii} &= -0·020\,113\,276\,P + 0·015\,421\,676\,Q - 0·026\,817\,889\,R
\end{aligned}
$$

where

$$P = +19{\cdot}8172 \qquad Q = -227{\cdot}7718 \qquad R = -394{\cdot}2509$$

We therefore have

$$\begin{aligned} \nu &= -0{\cdot}655\,352\,905 \pm 0{\cdot}065\,80 \\ \nu^{\text{i}} &= -0{\cdot}007\,003\,642 \pm 0{\cdot}005\,235 \\ \nu^{\text{ii}} &= +0{\cdot}066\,617\,652 \pm 0{\cdot}005\,698 \end{aligned}$$

with which the residuals in the normal equations (130) are

$$+0{\cdot}000\,001'' \qquad +0{\cdot}000\,002'' \qquad +0{\cdot}000\,002''$$

The residuals in the weighted observation equations (129) are

″	″	″
−0·1600	−0·5908	+0·1850
−0·3664	−0·4964	+0·0895
−0·6868	−0·9370	−0·6835
−0·6303	−0·0715	−0·0998
+0·3278	+0·0785	−0·3246
−0·3097	+0·1316	−0·4778
		−0·8186

and the probable error of any one of the weighted observation equations (129) is $\pm 0{\cdot}3479''$.

As a final check on the solution, we have the relation

$$[nn] + [an]\,\nu + [bn]\,\nu^{\text{i}} + [cn]\,\nu^{\text{ii}} - [vv] = 0$$

which is satisfied thus:

	″
$[nn]$	+41·913 63
$[an]\nu$	−12·987 26
$[bn]\nu^{\text{i}}$	+ 1·595 23
$[cn]\nu^{\text{ii}}$	−26·264 07
Sum	+ 4·257 53
$[vv]$	+ 4·257 31
Check	+ 0·000 22

It may be remarked that the weight factors were so chosen as to give a probable error of $\pm 0{\cdot}360''$ for each of the observation equations in the group (129), and the fact that this probable error comes out $\pm 0{\cdot}348''$ seems to indicate that the relative weights were sufficiently exact. The 9th and 19th observation equations give the largest residuals, and perhaps it might have been better to omit them, but it is not likely that their retention can have sensibly affected the corrections to the masses.

From the solution of the group of equations (129) we now have as the definitive result of this investigation

$$
\begin{aligned}
\text{Mass of Mercury} &= \frac{0.344\,647 \pm 0.065\,80}{3\,000\,000} = \frac{1}{8\,704\,559 \pm 1\,724\,742} \\
\text{Mass of Venus} &= \frac{0.992\,996 \pm 0.005\,235}{401\,847} = \frac{1}{404\,681 \pm 2\,134} \\
\text{Mass of Earth} &= \frac{1.066\,618 \pm 0.005\,698}{354\,936} = \frac{1}{332\,768 \pm 1\,778}
\end{aligned}
\qquad (131)
$$

Or, expressed decimally,

$$
\begin{aligned}
\text{Mass of Mercury} &= 0.000\,000\,114\,882 \pm 0.000\,000\,021\,933 \\
\text{Mass of Venus} &= 0.000\,002\,471\,082 \pm 0.000\,000\,013\,027 \\
\text{Mass of Earth} &= 0.000\,003\,005\,097 \pm 0.000\,000\,016\,056
\end{aligned}
\qquad (132)
$$

With respect to the data employed in these determinations, transits of Mercury have been used down to 1882, transits of Venus to 1883, meridian observations of Mercury to 1842, meridian observations of Venus to 1871, meridian observations of the Sun to 1850, and meridian observations of Mars to 1858. Since these dates there have accumulated 47 years of meridian observations upon Mercury, 18 years upon Venus, 39 years upon the Sun, and 31 years upon Mars; but to utilize them exhaustively for determining the masses of the three interior planets would necessitate an amount of labor almost equivalent to computing new tables of Mercury, Venus, the Sun, and Mars.

Some explanation seems desirable respecting the method of computing the probable errors of the planetary masses in (131) and (132). The expressions for these masses are of three forms, which may be written

$$m = \frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c} \qquad (133)$$

$$m = e \pm f \qquad (134)$$

$$m = \frac{1}{g \pm h} \qquad (135)$$

Whichever of these forms is employed, it is clear that when the probable error is added to, and subtracted from, the most probable value of the mass, the resulting limiting values should be the same. That condition is manifestly fulfilled by the forms (133) and (134), and in order that it may be fulfilled by the form (135) we must have

$$
\begin{aligned}
&(a+b)/c = 1/(g+h) \\
\text{and also} \quad &(a-b)/c = 1/(g-h)
\end{aligned}
\qquad (136)
$$

The form (135) is usually derived from (133) by the binomial theorem, thus

$$\frac{a \pm b}{c} = \frac{1}{\frac{c}{a}\left(1 \pm \frac{b}{a}\right)^{-1}} = \frac{1}{\frac{c}{a} \mp \frac{cb}{a^2} + \text{etc.}} \qquad (137)$$

all terms of the expansion beyond the second being neglected; and for that reason the formula so obtained is sufficiently exact only when b is small compared with a. To obtain a more general expression, we remark that as the maximum and minimum values of (133) are

$$\frac{1}{\dfrac{c}{a+b}} \quad \text{and} \quad \frac{1}{\dfrac{c}{a-b}}$$

we may write

$$2h = \frac{c}{a+b} - \frac{c}{a-b} = -\frac{2cb}{a^2-b^2}$$

and therefore, very approximately

$$\frac{a \pm b}{c} = \frac{1}{\dfrac{c}{a} \mp \dfrac{cb}{a^2-b^2}} \tag{138}$$

As c/a is not precisely equal to

$$\frac{1}{2}\left(\frac{c}{a+b} + \frac{c}{a-b}\right)$$

formula (138) does not rigorously fulfill the conditions (136), but it is far more exact than (137) when b is not small relatively to a, and as b diminishes the results given by (137) and (138) tend to become identical.

By reasoning similar to that employed in deducing formula (138), it is easy to obtain

$$e \pm f = \frac{1}{\dfrac{1}{e} \mp \dfrac{f}{e^2-f^2}} \tag{139}$$

which is required in passing from form (134) to (135). Also, for passing from (135) to (134)

$$\frac{1}{g \pm h} = \frac{1}{g} \mp \frac{h}{g^2-h^2} \tag{140}$$

But if h has been derived through the forms (138) or (139), as is usually the case, then according to (133) the probable error is $\pm b/c$, and by expressing that quantity in terms of g and h we find, with all needful accuracy

$$\frac{1}{g \pm h} = \frac{1}{g} \mp \frac{h}{g^2}\left(1 - \frac{h^2}{g^2}\right) \tag{141}$$

19.—TRIGONOMETRICAL DETERMINATIONS OF THE SOLAR PARALLAX.

Observations of Mars, when in opposition to the Sun, and at its least distance from the Earth, constitute one of the oldest trigonometrical methods of determining the solar parallax. There are two ways of making the observations. Either the planet is observed on or near the meridian, at two stations situated respectively, in the northern and southern hemispheres; or it is observed soon after rising, and just before setting, at a single station. The first method will be termed the meridian method, the second the diurnal method. In the meridian method the observations may be made either with a transit circle, or with a micrometer attached to an equatorial telescope. In the diurnal method they may be made either with an equatorial telescope, or with a heliometer.

The values of the solar parallax resulting from some of the most noteworthy attempts by the meridian method are as follows:

	″
1672. J. D. Cassini (196, p. 114)	9·5
1751. La Caille (Ephémérides des mouvements célestes depuis 1765 jusqu'en 1774. Paris. Introduction, p. 1)	10·38
1835. Henderson (224, p. 103)	9·028
1836. Taylor (265, p. 71)	9·253
1856. Gilliss and Gould (216, p. cclxxxviij)	8·495
1863. Winnecke (269, p. 264)	8·964
1865. E. J. Stone (252, p. 97)	8·943
1865. Asaph Hall (217, p. lxiv)	8·842
1867. Newcomb (232, p. 22)	8·855
1879. Downing (198, p. 127)	8·960
1881. Eastman (200, p. 41)	8·953
1882. E. J. Stone (264, p. 300)	8·95

The following are some of the results from the diurnal method:

	″
1672. J. D. Cassini (196, p. 107)	10·2
1672. Flamstead (209)	10
1719. Pound and Bradley (219, p. 114, and 243, p. 1111)	10·5
1857. W. C. Bond (195, p. 53)	8·605
1877. Maxwell Hall (218, p. 121)	8·789
1879. Gill (214, p. 163)	8·78

Owing to the comparative nearness of the asteroids, and their small, well-defined disks, it has been thought that the solar parallax might be accurately derived from observations made upon them in the manner just described for Mars. Several attempts in that direction are now in progress, but the following are believed to be all the results hitherto published:

	″
1875. Galle, from Flora (211, p. 7, and 213, p. 67)	8·873
1877. Lord Lindsay and Dr. Gill, from Juno (230, p. 211)	8·765

The same method has also been applied to Mercury and Venus, but there are great difficulties in the way of obtaining satisfactory results from these planets.

Transits of Venus.—Until comparatively recently, astronomers have believed that transits of Venus furnish by far the most accurate means of determining the solar parallax. Such transits have been observed by three different methods, namely: (1) by noting the times of contact between the limbs of Venus and the Sun; (2) by observing the position of Venus upon the Sun's disk with a heliometer; (3) by photographing the Sun with Venus upon its disk, and subsequently measuring the photographs.

Contact Observations.—The following are some of the results for solar parallax obtained by different astronomers from contact observations of the transits of Venus in 1761, 1769, 1874, and 1882:

Transit of 1761.	″
1763. HORNSBY (225, p. 494)	9·73
1763. SHORT (250, p. 340)	8·56
1765. PINGRÉ (239, p. 32)	10·10
1767. PLANMAN (242, p. 127)	8·49
Transit of 1769.	
1769. EULER (203½, p. 518)	8·80
1771. HORNSBY (226, p. 579)	8·78
1771. LA LANDE (227, p. 798)	8·62
1771. MASKELYNE (12, vol. 1, p. 413)	8·723
1772. LEXELL (229½, pp. 661 and 672)	8·63
1772. PINGRÉ (240, p. 419)	8·80
1772. PLANMAN (5, p. 407)	8·43
1786. DU SÉJOUR (199, p. 486)	8·851
1814. DELAMBRE (3, T. 1, p. xliv)	8·552
1815. FERRER (208, p. 286)	8·58
1865. POWALKY (244, p. 22)	8·832
1868. E. J. STONE (256, p. 264)	8·91
Transits of 1761 *and* 1769.	
1835. ENCKE (203, p. 309)	8·571
Transit of 1874.	
1877. AIRY, from British observations (193, p. 16)	8·754
1878. E. J. STONE, from British observations (261, p. 294)	8·884
1878. TUPMAN, from British observations (267, p. 455)	8·846
1881. PUISEUX, from French observations (247, p. 487)	8·93
1881. E. J. STONE, from French observations (263, p. 328)	8·88
Transit of 1882.	
1887. E. J. STONE, from British observations (251, p. 7)	8·832
1887. CRULS, from Brazilian observations (197, p. 1237)	8·808

The large differences in the parallaxes obtained by different astronomers from the same observations are due to the circumstance that, as the instants of contact are rendered uncertain by the intervention of various disturbing phenomena, many of the observers record two or three different times, corresponding to as many different phases which they endeavor to describe, and thus the resulting parallaxes are influenced to a certain extent by the interpretation put upon these descriptions. The interior contacts give better results than the exterior ones, but in any case the probable error is large. From 61 selected observations of interior contacts of the transit of 1874, discussed by Colonel TUPMAN,* the present writer found the probable error

*267, twenty on p. 450 and forty-one on p. 453.

of an observed time of contact to be ±4·59^s, which corresponds to a probable error of ±0·15″ in the distance between the centers of the Sun and Venus. Actual errors of from 20 to 30 seconds in the observed times of contacts are by no means uncommon.

Observations with Heliometers.—A few heliometers were used in observing the transits of 1874 and 1882, but until the resulting values of the solar parallax are published the accuracy of their work can not be satisfactorily estimated.

Photographic Observations.—For observing the transit of 1874 photography was extensively employed by the English, French, German, and United States parties, but the photoheliographs used by the English and Germans differed radically from those used by the French and Americans. In the subsequent measurement of the pictures the English and Germans failed to obtain satisfactory results, while the Americans, and apparently the French also, succeeded completely; and thus it came about that no photographs of the transit of 1882 were attempted either by the English or by the Germans, while the Americans and French took many hundreds. So far as known, the following are the values of the solar parallax yielded by the photographs:

Transit of 1874.

	″ ″
1881. TODD, from the United States photographs (266, p. 493)	8·883 ± 0·034
1885. OBRECHT, from the French daguerreotypes (237, p. 1121)	8·81 ± 0·06

Transit of 1882.

1888. HARKNESS, from the United States photographs	8·842 ± 0·012

It may be well to add that TODD's result depends upon 213 photographs, OBRECHT's upon 82 daguerreotypes, and that of the present writer upon 1475 photographs. The multiplication of the square roots of these numbers by the respective probable errors of the results gives ±0·496″, ±0·544″, and ±0·461″ for the probable error of a single picture.

Discussion of Results.—To facilitate the determination of a definitive value from the foregoing results, they have been re-arranged in Table IV, the construction of which will now be explained.

Of the many reductions of the observations of the transits of Venus in 1761 and 1769, all made prior to ENCKE's time are more or less incomplete, and will therefore be ignored. In dealing with the remaining reductions of contact observations it must be borne in mind that within certain limits the value obtained for the parallax depends upon the meaning attached by the computer to the records made by the various observers, and as these records will frequently bear more than one interpretation, the mean of the conclusions reached by several thoroughly competent computers must generally have a higher degree of probability than the conclusion of any one of them. Accordingly, the numbers entered in Table IV are, for the transits of 1761 and 1769, the mean of the results obtained by ENCKE, POWALKY, and E. J. STONE; for the transit of 1874, the mean of the five results obtained by AIRY, E. J. STONE, TUPMAN, and PUISEUX; and for the transit of 1882, the mean of the results obtained by E. J. STONE and CRULS, giving the former double weight.

TABLE IV.—*Values of the Solar Parallax obtained by Trigonometrical Methods.*

Transits of Venus, contact observations.	″
Transits of 1761 and 1769	8·771
Transit of 1874, English and French observations	8·859
Transit of 1882, English and Brazilian observations	8·824
Transits of Venus, photographic observations.	
Transit of 1874, United States and French photographs	8·859
Transit of 1882, United States photographs	8·842
Oppositions of Mars.	
Opposition of 1832:	
HENDERSON	9·028
TAYLOR	9·253
Opposition of 1849–'50:	
GILLISS and GOULD	8·495
W. C. BOND	8·605
Opposition of 1862:	
NEWCOMB	8·855
ASAPH HALL	8·842
Opposition of 1877:	
EASTMAN	8·953
GILL	8·78
MAXWELL HALL	8·789
Oppositions of Asteroids.	
Flora, in 1873. GALLE	8·873
Juno, in 1874. Lord LINDSAY and Dr. GILL	8·765

From the photographs of transits of Venus the results given are, for the transit of 1874 the mean of the values found by TODD and OBRECHT, giving the former double weight, and for the transit of 1882 the result found by the present writer for the U. S. Transit of Venus Commission.

The early observations of Mars for parallax have been ignored because they were made with insufficient instrumental appliances. With respect to the values entered in Table IV, for the opposition of 1832 there exist only the determinations by HENDERSON and TAYLOR, and for the opposition of 1849–'50 only the determinations by GILLISS and GOULD, and by W. C. BOND. For the opposition of 1862 the results obtained by E. J. STONE and WINNECKE rest upon but a small part of the data used by NEWCOMB, and therefore only the results obtained by NEWCOMB and A. HALL require consideration. Similarly, for the opposition of 1877 we have to deal only with the results obtained by EASTMAN, GILL, and M. HALL, because DOWNING employed a very small part of the data used by EASTMAN, and E. J. STONE's paper is virtually an indorsement of EASTMAN's result.

It is believed that the numbers in Table IV fairly represent all the material now in existence for the trigonometrical determination of the solar parallax. What is the most probable result that can be obtained from them? The arithmetical mean of all the values gives

$$p = 8{\cdot}837'' \pm 0{\cdot}0614'' \qquad (142)$$

The means of the results from observations of Mars are, from the oppositions of 1832 and 1849–'50,

$$p = 8.845''$$

and from the oppositions of 1862 and 1877

$$p = 8.844''$$

Nevertheless, the four values resulting from the oppositions of 1832 and 1849–'50 are so discordant that they should probably be rejected. Doing so, the arithmetical mean of all the other values in Table IV gives

$$p = 8.834'' \pm 0.0086'' \tag{143}$$

Again, taking the means according to the methods of observation, we obtain

From transits of Venus, contacts	$p = 8.818''$
From transits of Venus, photographs	$p = 8.850$
From Mars	$p = 8.844$
From Asteroids	$p = 8.819$

and the arithmetical mean is

$$p = 8.833'' \pm 0.0056'' \tag{144}$$

Finally, considering only the results which seem most likely to be free from constant errors, we have

From photographs of transits of Venus, 1874 . .	$p = 8.859''$
From photographs of transits of Venus, 1882 . .	$p = 8.842$
From Gill's observations of Mars	$p = 8.780$
From Galle's observations of Flora	$p = 8.873$
From Gill's observations of Juno	$p = 8.765$

and the arithmetical mean is

$$p = 8.824'' \pm 0.0146'' \tag{145}$$

Except in the magnitude of their probable errors, these four means scarcely differ from each other: but so far as there is any choice among them, well settled principles would lead to the selection of (143), and accordingly that will be adopted.

20.—GENERAL FORMS OF THE CONDITIONAL EQUATIONS.

If l is the length of a simple pendulum which makes one vibration per second of mean solar time, the observed force of gravity will be

$$g = \pi^2 l \tag{146}$$

Upon the assumption that the Earth attracts as if its entire mass were concentrated at its center of gravity, its attraction at a point upon its surface in latitude φ will be

$$k^2 \mathrm{E}' / a^2 \rho^2 \tag{147}$$

The observed force of gravity is the Earth's attractive force diminished by the resolved value of its centrifugal force. Putting σ for the ratio of the centrifugal force to the force of gravity at the geographical latitude φ, we have*

$$\sigma = \frac{4\pi^2 \mathrm{N} \cos \varphi}{g t_1^2} = \frac{4 \mathrm{N} \cos \varphi}{l t_1^2}$$

where N cos φ is the radius of the Earth at latitude φ, and t_1 is the number of mean solar seconds in a sidereal day. But†

$$\mathrm{N} = \frac{a}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

whence the centrifugal force at latitude φ is

$$\sigma g = \frac{4ga \cos \varphi}{l t_1^2 \sqrt{(1 - e^2 \sin^2 \varphi)}}$$

and the resolved part of that force acting in the direction of the vertical is

$$\sigma g \cos \varphi = \frac{4ga \cos^2 \varphi}{l t_1^2 \sqrt{(1 - e^2 \sin^2 \varphi)}} \tag{148}$$

Equating the Earth's attraction to the force of gravity augmented by the centrifugal force, we have

$$\frac{k^2 \mathrm{E}'}{a^2 \rho^2} = g(1 + \sigma \cos \varphi)$$

Whence, by (146)

$$\frac{k^2}{\pi^2} = \frac{a^2 \rho^2 l}{\mathrm{E}'} (1 + \sigma \cos \varphi) \tag{149}$$

*Compare 14, 2 Teil, p. 84.

†28, p. 323.

If T is the length of the sidereal year, expressed in seconds of mean solar time, and a_1 is that value of the semi-major axis of the Earth's orbit which would satisfy KEPLER'S third law, then

$$T^2 = \frac{4\pi^2 a_1^3}{k^2(S + E')} \tag{150}$$

and by eliminating k^2/π^2 between (149) and (150), and re-arranging the terms

$$\frac{S + E'}{E'} = \frac{4a_1^3}{lT^2a^2\rho^2(1 + \sigma \cos \varphi)} \tag{151}$$

Owing to the spheroidal form of the Earth, those points which have $\sqrt{\frac{1}{3}}$ for the sine of their latitude are the only ones upon the Earth's surface at which a pendulum will vibrate as it would if the mass of the Earth were concentrated at its center of gravity. In order to justify the assumption made in equation (147), we must therefore take $\sin^2 \varphi = \frac{1}{3}$, and consequently $\cos^2 \varphi = \frac{2}{3}$. We also put

$$a = r \sin p$$
$$a_1 = r\,(1 + \varkappa)$$

Substituting these values in (148) and (151), they become

$$\frac{S + E'}{E'} = \frac{4a(1 + \varkappa)^3}{l\rho^2T^2 \sin^3 p(1 + \sigma\sqrt{\frac{2}{3}})} \tag{152}$$

$$\sigma\sqrt{\tfrac{2}{3}} = \frac{8a}{3lt_1^2\sqrt{(1 - \frac{1}{3}e^2)}} \tag{153}$$

From (3), (5), (8), (14), and (17) we have

$a = 20\,926\,202$ feet $l = 3{\cdot}256\,872$ feet
$e^2 = 0{\cdot}006\,803\,481\,019$ $t_1^2 = 7\,424\,252\,068^s$
$\rho^2 = 0{\cdot}997\,742\,482$

Substituting these values in (152) and (153)

$$\sigma\sqrt{\tfrac{2}{3}} = 0{\cdot}002\,310\,461\,5$$

$$\frac{S + E'}{E'} = [7{\cdot}409\,929\,0]\frac{(1 + \varkappa)^3}{T^2 \sin^3 p} \tag{154}$$

where the quantity in brackets is the logarithm of the number which it represents.

By attributing proper values to the symbols in equation (154) it may be applied either to the Earth revolving around the Sun or to the Moon revolving around the Earth. In the former case we shall have from (16) and (25)

$$T = 31\,558\,149{\cdot}314^s$$
$$(1 + \varkappa) = 0{\cdot}999\,998\,710$$

and the substitution of these values in (154) gives

$$p = [2{\cdot}784\,993\,2]\sqrt[3]{\left(\frac{E'}{S + E'}\right)} \tag{155}$$

where p is expressed in seconds of arc. Of the constants which enter this expression l is the most uncertain, and as it is not trustworthy beyond the fifth significant figure, the logarithmic coefficient in (155) is not trustworthy beyond the fifth decimal place.

The Earth's mass is very small compared with that of the Sun, and if in conformity with custom we take the latter for unity, we may write with all needful accuracy

$$\frac{E'}{S+E'}=E'$$

Further, in (155) the symbol E' refers to the mass of the Earth alone, while the quantity usually called the mass of the Earth is the combined mass of the Earth and Moon, denoted by the symbol E. Bearing in mind that the mass of the Moon is expressed in terms of the Earth's mass as unity, it is evident that

$$E'=\frac{E'(1+M)}{1+M}=\frac{E}{1+M}$$

and therefore when E' is changed into E (155) takes the form

$$p=[2{\cdot}784\,993\,2]\sqrt[3]{\left(\frac{E}{1+M}\right)} \qquad (156)$$

In order to apply (154) to the case of the Moon revolving around the Earth we must change the symbol S into M, $(1+\varkappa)$ into $(1+\varkappa')$, T into T_1, and p into P Then, from (39) and (57)

$$T_1=2\,360\,591{\cdot}5^s$$
$$(1+\varkappa')=1{\cdot}000\,908\,743$$

and by substituting these values we find, after a slight transformation

$$\frac{E'}{M}=\frac{\sin^3 P}{[4{\cdot}665\,070\,7-10]-\sin^3 P} \qquad (157)$$

where, as in (155), the logarithmic term is not trustworthy beyond the fifth place of decimals.

In 1755 D'Alembert determined the Moon's mass from the phenomena of precession and nutation, but to do this with extreme accuracy seems a difficult matter. The most recent attempt is by Mr. E. J. Stone,* who states that his equations include all terms of the third order in the lunar theory. With some changes of notation, and after restoring the factor cos ω_0, they are

$$\varepsilon=Mr^3/Sr_1^3$$
$$\mathfrak{L}=(A\varkappa+B\varkappa\varepsilon)\cos\omega_0 \qquad (158)$$
$$\mathfrak{N}=C\varkappa\varepsilon\cos\omega_0$$

*187, p. 43.

where $\varkappa$ is a constant depending upon the Sun's mean disturbing force, the moments of inertia of the Earth, and the Earth's angular velocity; and

$$\begin{aligned} A &= 1 + \frac{3}{2}e_1^2 \\ B &= 1 + \frac{3}{2}e_2^2 - 6\gamma^2 \\ C &= \frac{2\gamma}{\mu}\left(1 + \frac{3}{2}e_2^2 - \frac{5}{2}\gamma^2\right) \end{aligned} \tag{159}$$

As Mr. STONE has not published any details respecting these formulæ, it may be well to show how they can be derived. According to SERRET*

$$\begin{aligned} A &= 1 + \frac{3}{2}e_1^2 \\ B &= 1 + \frac{3}{2}e_2^2 - \frac{3}{2}I^2 \\ C &= \frac{I}{\mu} \end{aligned} \tag{160}$$

but that geometer has neglected terms of the third order with respect to the inclination and eccentricity of the Moon's orbit, and to restore them we must replace I by $(1 + \frac{3}{2}e_2^2)\sin I \cos I$. Bearing in mind that $\sin\frac{1}{2}I = \gamma$, we have

$$\begin{aligned} \sin I \cos I &= \sin 2(\tfrac{1}{2}I) \cos 2(\tfrac{1}{2}I) \\ &= 2\gamma(1-\gamma^2)^{\frac{1}{2}}(1-2\gamma^2) \\ &= (2\gamma - 4\gamma^3)(1-\gamma^2)^{\frac{1}{2}} = 2\gamma - 5\gamma^3 - \text{etc.} \end{aligned}$$

and therefore to take account of all terms of the third order in the equations (160), it suffices to replace I by

$$\left(1 + \frac{3}{2}e_2^2\right)(2\gamma - 5\gamma^3) = 2\gamma + 3e_2^2\gamma - 5\gamma^3 \tag{161}$$

and I^2 by $4\gamma^2$. Upon making the substitution the equations (159) result.

Reverting to the equations (158), eliminating $\varkappa$ and ε from them, and introducing the sines of the parallaxes instead of the mean distances, we get

$$M = \frac{A\mathfrak{A}S \sin^3 p}{(C\mathfrak{L} - B\mathfrak{A}) \sin^3 P} \tag{162}$$

But from (154)

$$S \sin^3 p = E'([2{\cdot}411\ 704\ 2 - 10] - \sin^3 p)$$

which being substituted in (162) gives

$$\frac{E'}{M} = \frac{(C\mathfrak{L} - B\mathfrak{A}) \sin^3 P}{A\mathfrak{A}([2{\cdot}411\ 704\ 2 - 10] - \sin^3 p)} \tag{163}$$

*83, pp. 303, 313, and 315.

From (24), (61), (64), and (52)

$$e_1 = 0.016\,771\,049 \qquad \gamma = 0.044\,886\,793$$
$$e_2 = 0.054\,899\,720 \qquad \mu = 0.337\,815\,984$$

and with these values, from (159)

$$\begin{aligned} A &= +1.000\,421\,902 \\ B &= +0.992\,432\,024 \\ C &= +0.265\,609\,855 \end{aligned} \tag{164}$$

In (163) the term $\sin^3 p$ is so small that we may safely use the value found by assuming $p = 8.834''$, and by substituting that, together with the values of A, B, and C from (164), we shall have

$$\frac{E'}{M} = \sin^3 P \left\{ 10\,288\,642 \frac{\mathfrak{L}}{\mathfrak{A}} - 38\,442\,769 \right\} \tag{165}$$

To find the relations existing between $\mathfrak{A}$, $\mathfrak{L}$, and P, we equate the right hand members of (157) and (165), and thus obtain

$$\mathfrak{A} = \mathfrak{L} \left\{ \frac{1 - 216\,236.65 \sin^3 P}{3.757\,444\,9 - 807\,952.64 \sin^3 P} \right\} \tag{166}$$

The parallactic inequality of the Moon is given by the expression*

$$\sin Q' = F \frac{E' - M}{E' + M} \times \frac{a_2}{a_1} \sin D \tag{167}$$

where D is the mean angular distance of the Moon from the Sun; and when sin D becomes unity, Q′ becomes Q. But

$$a_2 = \frac{a(1 + \varkappa')}{\sin P} \qquad a_1 = \frac{a(1 + \varkappa)}{\sin p}$$

Whence, with the numerical values of $(1 + \varkappa')$ and $(1 + \varkappa)$ from (57) and (25)

$$\frac{a_2}{a_1} = \frac{(1 + \varkappa')\sin p}{(1 + \varkappa)\sin P} = 1.000\,910\,034 \frac{\sin p}{\sin P} \tag{168}$$

Delaunay gives†

$$\frac{F}{\text{arc } 1''} \times \frac{a_2}{a_1} = 127.242\,3'' \tag{169}$$

*52, T. 2, p. 847, eq. 342; 53, p. 37, and 57, p. 36.

†52, T. 2, p. 847, eq. 342, and 53, p. 18.

But instead of using the rigorous formula (168), he arranged his numerical computations so as to employ the expression

$$\frac{a_2}{a_1} = \frac{p(1 + \varkappa')}{P} \tag{170}$$

from which, with

$$p = 8{\cdot}75'' \qquad P = 3\,422{\cdot}7'' \qquad 1 + \varkappa' = 1{\cdot}000\,908\,743$$

he obtained

$$a_2/a_1 = 0{\cdot}002\,558\,784$$

Substituting that value in (169), we find

$$F = 0{\cdot}241\,086 \tag{171}$$

From the data given by Professor NEWCOMB,* it appears that the value of F implicitly contained in HANSEN'S lunar tables is

$$F = 122{\cdot}032'' \text{ arc } 1'' \frac{E' + M}{E' - M} \times \frac{a_1}{a_2} \tag{172}$$

but it is not quite clear whether a_1/a_2 should be derived from (168) or (170). If the former, as according to NEWCOMB, HANSEN used

$$p = 8{\cdot}608\,5'' \qquad P = 3\,422{\cdot}25'' \qquad M = \frac{1}{80}$$

(172) gives

$$F = 0{\cdot}240\,922$$

If the latter

$$F = 0{\cdot}240\,933 \tag{173}$$

The mean of (171) and (173) will be adopted, namely

$$F = 0{\cdot}241\,010 \tag{174}$$

Reverting to (167), making sin D unity, substituting the value of a_2/a_1 from (168), that of F from (174), and re-arranging the terms, we obtain

$$p = [5{\cdot}303\,124\,8 - 10]\,PQ\frac{E' + M}{E' - M} \tag{175}$$

*62, pp. 69, 79, and 87.

where the quantity within the brackets is the logarithm of the number which it represents, and on account of the uncertainty in F it is not trustworthy beyond the fifth place of decimals.

The lunar equation of the Earth's motion is*

$$\delta\nu = -\frac{M}{E' + M} \times \frac{\sin p'}{\sin P'} \times \cos s' \sin(\nu' - \nu) \tag{176}$$

in which p' and P' are the actual values of the solar and lunar parallaxes at the instant for which $\delta\nu$ is required. For any given lunation, $\delta\nu$ will evidently attain its maximum value when $\sin(\nu' - \nu) = 1$, that is, when the longitudes of the Sun and Moon differ by ninety degrees. If now we have an extensive series of observed values of $\delta\nu$, covering many complete revolutions of the Moon's node, $\delta\nu$ will have assumed all possible values, the mean of which will be the constant of the lunar inequality; p' will have assumed all possible values, the mean of which will be the constant of solar parallax; and the Moon will have had all possible latitudes, the mean of which will be zero. With P' the case is somewhat different. It is equal to the constant of lunar parallax, plus a series of terms multiplied by factors made up of the mean anomaly of the Sun, the mean anomaly of the Moon, the mean distance of the Moon from its ascending node, and the difference of the mean longitudes of the Sun and Moon. All these terms, except those involving the difference of the mean longitudes, will assume all possible values and vanish from the mean. The mean of all the values of P' will therefore be, $P +$ terms depending upon the difference of mean longitudes† of the Sun and Moon. Turning now to the lunar theory, we find but a single term of this kind,‡ and its value is $28{\cdot}178\,8'' \cos 2D$. As we have supposed all our observations of $\delta\nu$ to be made when D was 90°, the value of this term will be $-28{\cdot}18''$, and the mean value of P' will be $P - 28{\cdot}18''$. However, it will be better to write

$$P' = P(1 + G) \tag{177}$$

and then, according to Delaunay, we shall have‡

$$PG = -28{\cdot}178\,8''$$

which, with $P = 3\,422{\cdot}7''$, gives

$$G = -0{\cdot}008\,232\,92 \tag{178}$$

To find the value of G employed in Hansen's tables of the Moon we have, according to Newcomb§

$$PG = -28{\cdot}225''$$

which, with $P = 3\,422{\cdot}25''$, gives

$$G = -0{\cdot}008\,247\,50 \tag{179}$$

*41, p. 47. †Strictly speaking, it should be the difference of *true* longitudes of the Sun and Moon.
‡52, T. 2, p. 917, eq. (27). §62, p. 105.

The substitution of the mean of (178) and (179) in (177) gives

$$P' = 0.991\,759\,8\,P$$

Substituting the mean values thus found in (176), and re-arranging the terms, we obtain

$$p = [4.681\,962\,4 - 10]\,PL\frac{E' + M}{M} \tag{180}$$

where the logarithm of the numerical coefficient is trustworthy only to five places of decimals.

If V is the velocity of light, θ the light equation, or, in other words, the time required by light to traverse the mean radius of the Earth's orbit, and a the equatorial semidiameter of the Earth, then

$$\sin p = \frac{a}{V\theta} \tag{181}$$

whence, with the value of a from (3)

$$p = \frac{[8.912\,481\,6]}{V\theta} \tag{182}$$

The mean velocity of the Earth in its orbit is*

$$\frac{2\pi r}{T\sqrt{(1 - e_1^2)}}$$

and if we assume the constant of aberration to be the ratio of that velocity to the velocity of light, then

$$\tan \alpha = \frac{2\pi r}{VT\sqrt{(1 - e_1^2)}}$$

But $r = a/\sin p$, whence

$$p = \frac{2\pi a}{TV\alpha \operatorname{arc}^2 1''\sqrt{(1 - e_1^2)}} \tag{183}$$

Substituting the values of a, T, and e_1 from (3), (16), and (24)

$$p = \frac{[7.526\,036\,2]}{V\alpha} \tag{184}$$

*2, vol. 1, p. 637.

21.—THE LEAST SQUARE ADJUSTMENT.

The conditions which must be satisfied by the adjusted quantities are embodied in equations (156), (157), (166), (175), (180), (182), and (184). For convenience of reference they are collected in the group (185), where the quantities within brackets are the logarithms of the numbers which they represent, and although given to six places of decimals they are trustworthy only to five.

$$
\begin{aligned}
\frac{1}{M} &= \frac{\sin^3 P}{[4{\cdot}665\,070\,70-10]-\sin^3 P} \\
0 \text{ or } v_1 &= p-[2{\cdot}784\,993]\left(\frac{E}{1+M}\right)^{\frac{1}{3}} \\
0 \text{ or } v_2 &= p-[5{\cdot}303\,125-10]\,PQ\,\frac{1+M}{1-M} \\
0 \text{ or } v_3 &= p-[4{\cdot}681\,962-10]\,PL\,\frac{1+M}{M} \\
0 \text{ or } v_4 &= p-\frac{[8{\cdot}912\,482]}{V\theta} \\
0 \text{ or } v_5 &= p-\frac{[7{\cdot}526\,036]}{V\alpha} \\
0 \text{ or } v_6 &= \mathfrak{A}-\mathfrak{L}\left\{\frac{1-216\,236{\cdot}65\sin^3 P}{3{\cdot}757\,444\,9-807\,952{\cdot}64\sin^3 P}\right\}
\end{aligned}
\qquad (185)
$$

If the observed quantities were rigorously exact, their substitution in the conditional equations (185) would reduce all the right hand members of the latter to zero; but in general this will not happen, and instead we shall obtain a series of residuals which may be designated v_1, v_2, v_3, etc., as indicated in (185). To make these residuals disappear, a series of corrections to the observed quantities must be determined, such that

$$
p=p'+dp \qquad P=P'+dP \qquad \mathfrak{L}=\mathfrak{L}'+d\mathfrak{L} \qquad \text{etc., etc.}
$$

where the observed quantities are distinguished by an accent, and the differentials are the required corrections. The first step will be to differentiate the equations (185), thus:

$$
\begin{aligned}
dM &= -3\cot P \operatorname{arc} 1''(1+M)\,dP \\
0 &= dp+\frac{p}{3(1+M)}dM-\frac{[8{\cdot}354\,98]}{3p^2(1+M)}dE \\
0 &= dp-\frac{p+[5{\cdot}303\,12]\,PQ}{1-M}dM-[5{\cdot}303\,12]\frac{1+M}{1-M}(P.dQ+Q.dP) \\
0 &= dp+\frac{p-[4{\cdot}681\,96]\,PL}{M}dM-[4{\cdot}681\,96]\frac{1+M}{M}(P.dL+L.dP) \\
0 &= dp+\frac{p}{V}dV+\frac{p}{\theta}d\theta \\
0 &= dp+\frac{p}{V}dV+\frac{p}{\alpha}d\alpha \\
0 &= d\mathfrak{A}-\frac{\mathfrak{A}}{\mathfrak{L}}d\mathfrak{L} \\
&\quad +3\operatorname{arc} 1''([5{\cdot}334\,93]\,\mathfrak{L}-[5{\cdot}907\,39]\,\mathfrak{A})\frac{\sin^2 P\cos P.dP}{[0{\cdot}574\,89]-[5{\cdot}907\,39\sin^3 P]}
\end{aligned}
\qquad (186)
$$

Upon reducing the coefficients in (186) to numbers by means of the data in (195), eliminating dM, which is not an observed quantity, and adding v_1, v_2, v_3, etc., to the resulting expressions, we obtain

$$\begin{aligned}
0 &= v_1 + dp - [7.4117 - 10]\, dP - [5.9802]\, dE \\
0 &= v_2 + dp + [8.1170 - 10]\, dP - [8.8478 - 10]\, dQ \\
0 &= v_3 + dp - [9.8134 - 10]\, dP - [0.1445]\, dL \\
0 &= v_4 + dp + [5.6758 - 10]\, dV + [8.2498 - 10]\, d\theta \\
0 &= v_5 + dp + [5.6758 - 10]\, dV + [9.6351 - 10]\, d\alpha \\
0 &= v_6 + d\mathfrak{N} - [9.2633 - 10]\, d\mathfrak{L} + [9.3243 - 10]\, dP
\end{aligned} \qquad (187)$$

From (185) and (195) the reciprocal of M was found to be 83·748, and accordingly that value was used in forming the differential equations (187). The final solution of these equations gives $1/M = 81.197$, but an inspection of the equations (186) shows that the difference between these two values can not sensibly affect any of the equations (187) except the third, in which it would have been better to have used $1/M = \frac{1}{2}(83.748 + 81.197) = 82.472$.

The equations (187) contain as unknown quantities ten corrections, which must be so determined as to satisfy these equations rigorously, and at the same time make the sum of the weighted squares of the corrections a minimum. To accomplish that we may use either the method of correlatives, or the method of double elimination, sometimes called the method of independent unknowns. As the probable errors of all the corrections are required, and the equations lend themselves readily to a double elimination, that method has been adopted. The equations (187) have been used to eliminate six of the unknowns, and in (189) each of the ten is expressed in terms of the remaining four, namely, in terms of dp, $d\mathfrak{L}$, $d\mathfrak{N}$, $d\alpha$, and known quantities; the coefficients there given being the logarithms of the numbers which they represent. The weights have been computed from the probable errors in (195) by means of the formula

$$\text{Log. weight} = -4 - 2\ (\text{log. probable error.}) \qquad (188)$$

Equations No. (189).

	dp	$d\mathfrak{L}$	$d\mathfrak{N}$	$d\alpha$	n	Log. weight.
$dp =$	+ 0·0000	. . . .	. . . .	. . . .	. . . .	0·1310
$dP =$	. . . .	+ 9·9390 — 10	— 0.6757	. . . .	$-[0.6757]\, v_6$	7·8344 — 10
$d\mathfrak{L} =$	. . . .	+ 0·0000	. . . .	. . . .	. . . .	1·2110
$d\mathfrak{N} =$	. . . . .	. . . .	+ 0·0000	. . . .	. . . .	9·9016 — 10
$dQ =$	+ 1·1522	+ 9·2082 — 10	— 9·9449 — 10	. . . .	$+[1.1522]\, v_2 \quad -[9.9449 - 10]\, v_6$	6·9118 — 10
$dL =$	+ 9·8555 — 10	— 9·6079 — 10	+ 0·3446	. . . .	$+[9.8555 - 10]\, v_3 + [0.3446]\, v_6$	9·5918 — 10
$d\alpha =$	. . . .	. . . .	. . . .	+ 0·0000	. . . .	9·9172 — 10
$d\theta =$	. . . .	. . . .	. . . .	+ 1·3853	$-[1.7502]\,(v_4 - v_5)$	5·9828 — 10
$dV =$	— 4·3242	. . . .	. . . .	— 3·9593	$-[4.3242]\, v_5$	2·8874 — 10
$dE =$	+ 4·0198 — 10	— 1·3705 — 10	+ 2·1072 — 10	. . . .	$+[4.0198 - 10]\, v_1 + [2.1072 - 10]\, v_6$	11·5888

Imagining the symbols dp, dP, $d\mathfrak{L}$, etc., in the first column of the equations (189) to be replaced by zeros, the weighted normal equations (190) have been formed in the usual way, but for lack of space on the page their absolute terms are represented by the letters A, B, C, D, whose values are given in (191).

$$\begin{array}{l} 0 = +A + 36{\cdot}482 \; dp - 0{\cdot}112\,63\, d\mathfrak{L} + 0{\cdot}614\,29\, d\mathfrak{N} + 14{\cdot}822 \; d\alpha \\ 0 = +B - 0{\cdot}112\,63\, dp + 16{\cdot}324 \; d\mathfrak{L} - 0{\cdot}378\,44\, d\mathfrak{N} \quad 0{\cdot}000\,00\, d\alpha \\ 0 = +C + 0{\cdot}614\,29\, dp - 0{\cdot}378\,44\, d\mathfrak{L} + 2{\cdot}861\,2 \; d\mathfrak{N} \quad 0{\cdot}000\,00\, d\alpha \\ 0 = +D + 14{\cdot}822 \; dp \quad 0{\cdot}000\,00\, d\mathfrak{L} \quad 0{\cdot}000\,00\, d\mathfrak{N} + 7{\cdot}2804 \; d\alpha \end{array} \qquad (190)$$

$$\begin{array}{l} A = +0{\cdot}425\,01 \; v_1 + 0{\cdot}164\,51 \; v_2 + 0{\cdot}200\,82\, v_3 + 34{\cdot}340 \; v_5 + 0{\cdot}614\,29\, v_6 \\ B = -0{\cdot}000\,953\,02\, v_1 + 0{\cdot}001\,871\,5\, v_2 - 0{\cdot}113\,55\, v_3 - 0{\cdot}378\,44\, v_6 \\ C = +0{\cdot}005\,197\,6 \; v_1 - 0{\cdot}010\,207 \; v_2 + 0{\cdot}619\,30\, v_3 + 2{\cdot}064\,0 \; v_6 \\ D = -0{\cdot}131\,31 \; v_4 + 14{\cdot}953 \; v_5 \end{array} \qquad (191)$$

The general solution of (190) is

$$\begin{array}{l} dp = -0{\cdot}161\,962\,A - 0{\cdot}000\,312\,B + 0{\cdot}034\,731\,C + 0{\cdot}329\,735\,D \\ d\mathfrak{L} = -0{\cdot}000\,312\,A - 0{\cdot}061\,449\,B - 0{\cdot}008\,061\,C + 0{\cdot}000\,636\,D \\ d\mathfrak{N} = +0{\cdot}034\,731\,A - 0{\cdot}008\,061\,B - 0{\cdot}358\,027\,C - 0{\cdot}070\,709\,D \\ d\alpha = +0{\cdot}329\,735\,A + 0{\cdot}000\,636\,B - 0{\cdot}070\,709\,C - 0{\cdot}808\,655\,D \end{array} \qquad (192)$$

By first substituting in (192) the values of A, B, C, and D from (191), and then substituting in (189) the resulting values of dp, $d\mathfrak{L}$, $d\mathfrak{N}$, and $d\alpha$, we obtain the formulæ (193), which are the expressions for the desired corrections to the observed values of p, P, $\mathfrak{L}$, etc., in terms of v_1, v_2, v_3, etc. The coefficients in (193) are the logarithms of the numbers which they represent.

Formulæ No. (193).

	v_1	v_2	v_3	v_4	v_5	v_6
dp =	− 8·8367 − 10	− 8·4313 − 10	− 8·0406 − 10	− 8·6365 − 10	− 9·8002 − 10	− 8·4423 − 10
dP =	− 8·7873 − 10	− 8·6474 − 10	+ 0·0065	− 8·6442 − 10	− 9·8079 − 10	− 0·1294
$d\mathfrak{L}$ =	− 6·0641 − 10	− 5·9245 − 10	+ 7·2840 − 10	− 5·9217 − 10	− 7·0806 − 10	+ 7·8079 − 10
$d\mathfrak{N}$ =	+ 8·1109 − 10	+ 7·9710 − 10	− 9·3301 − 10	+ 7·9678 − 10	+ 9·1315 − 10	− 9·8540 − 10
dQ =	− 9·9939 − 10	+ 1·1400	+ 8·5155 − 10	− 9·7944 − 10	− 0·9581	− 9·8085 − 10
dL =	− 8·3146 − 10	+ 7·1326 − 10	+ 9·3720 − 10	− 8·0203 − 10	− 9·1842 − 10	+ 9·7844 − 10
da =	+ 9·1454 − 10	+ 8·7401 − 10	+ 8·3494 − 10	+ 9·0260 − 10	− 9·8858 − 10	+ 8·7510 − 10
$d\theta$ =	+ 0·5307	+ 0·1254	+ 9·7347 − 10	− 1·7298	+ 1·5751	+ 0·1363
dV =	+ 2·2446	+ 1·8393	+ 1·4488	− 1·7278	− 2·8919	+ 1·8506
dE =	+ 3·9890 − 10	− 2·4493 − 10	− 2·1533 − 10	− 2·6551 − 10	− 3·8188 − 10	− 2·4038 − 10

For computing the probable errors of the adjusted values of p, P, $\mathfrak{L}$, etc., we shall need a series of formulæ expressing each of these adjusted values in terms of the originally observed values. As a first step toward finding them we substitute in (193) the values of v_1, v_2, v_3, etc., from (187), and thus obtain the equations (194).

Equations No. (194).

Factors.	Logarithmic coefficients for computing—				
	dp	dP	$d\mathfrak{P}$	$d\mathfrak{N}$	dQ
dp	+ 9·8928 — 10	— 9·3480 — 10	— 6·6395 — 10	+ 8·6715 — 10	— 0·4980
dP	— 7·0515 — 10	+ 9·9755 — 10	— 6·0154 — 10	+ 8·0611 — 10	— 8·4178 — 10
$d\mathfrak{P}$	— 7·7056 — 10	— 9·3927 — 10	+ 7·0712 — 10	— 9·1173 — 10	— 9·0718 — 10
$d\mathfrak{N}$	+ 8·4423 — 10	+ 0·1294	— 7·8079 — 10	+ 9·8540 — 10	+ 9·8085 — 10
dQ	— 7·2791 — 10	7·4952 — 10	— 4·7723 — 10	+ 6·8188 — 10	+ 9·9878 — 10
dL	— 8·1851 — 10	+ 0·1510	+ 7·4285 — 10	— 9·4746 — 10	+ 8·6600 — 10
da	+ 9·4353 — 10	+ 9·4436 — 10	+ 6·7163 — 10	— 8·7672 — 10	+ 0·5938
$d\theta$	+ 6·8863 — 10	+ 6·8940 — 10	+ 4·1715 — 10	— 6·2176 — 10	+ 8·0442 — 10
dV	+ 5·5051 — 10	+ 5·5125 — 10	+ 2·7855 — 10	— 4·8361 — 10	+ 6·6627 — 10
dE	— 4·8169	— 4·7675	— 2·0443	+ 4·0911	— 5·9741

Factors.	Logarithmic coefficients for computing—				
	dL	da	$d\theta$	dV	dE
dp	— 8·7237 — 10	+ 9·6488 — 10	+ 1·0341	+ 2·7484	— 3·3587 — 10
dP	+ 8·3940 — 10	+ 7·3612 — 10	+ 8·7462 — 10	+ 0·4592	— 1·0131 — 10
$d\mathfrak{P}$	+ 9·0477 — 10	+ 8·0143 — 10	+ 9·3996 — 10	+ 1·1139	— 1·6671 — 10
$d\mathfrak{N}$	— 9·7844 — 10	— 8·7510 — 10	— 0·1363	— 1·8506	+ 2·4038 — 10
dQ	+ 5·9804 — 10	+ 7·5879 — 10	+ 8·9732 — 10	+ 0·6871	— 1·2971 — 10
dL	+ 9·5165 — 10	+ 8·4939 — 10	+ 9·8792 — 10	+ 1·5933	— 2·2978 — 10
da	+ 8·8199 — 10	+ 9·5215 — 10	— 1·2108	+ 2·5276	+ 3·4545 — 10
$d\theta$	+ 6·2701 — 10	— 7·2758 — 10	+ 9·9796 — 10	+ 9·9776 — 10	+ 0·9049 — 10
dV	+ 4·8888 — 10	+ 5·4971 — 10	+ 6·8822 — 10	+ 8·5965 — 10	+ 9·5234 — 20
dE	— 4·2948	+ 5·1256	+ 6·5109	+ 8·2248	+ 9·9692 — 10

$$dM = +[6{\cdot}948\,04 - 10]\,dP$$

The tabular form has been adopted to save space. It is to be read thus

$$dp = +[9{\cdot}8928 - 10]\,dp - [7{\cdot}0515 - 10]\,dP - [7{\cdot}7056 - 10]\,d\mathfrak{P} + [8{\cdot}4423 - 10]\,d\mathfrak{N} - [7{\cdot}2791 - 10]\,dQ - [8{\cdot}1851 - 10]\,dL + [9{\cdot}4353 - 10]\,d\alpha + [6{\cdot}8863 - 10]\,d\theta + [5{\cdot}5051 - 10]\,dV - [4{\cdot}8169]\,dE$$

and similarly for the other quantities. The further consideration of these equations is relegated to page 71.

22.—NUMERICAL VALUES OF THE CORRECTIONS BY ADJUSTMENT.

The adopted values of the observed quantities arrived at in the preceding pages are marked numbers (66), (67), (75), (89), (90), (91), (92), (101), (132), and (143). For convenience of reference they are all collected in the group (195), where those which vary with the time are reduced to the epoch 1850·0.

$$
\begin{aligned}
p &= 8{\cdot}834'' \pm 0{\cdot}008\,6'' \\
P &= 3\,423{\cdot}08'' \pm 0{\cdot}121'' \\
\mathfrak{L} &= 50{\cdot}358\,6'' \pm 0{\cdot}002\,48'' \\
\mathfrak{N} &= 9{\cdot}233\,1'' \pm 0{\cdot}011\,2'' \\
Q &= 125{\cdot}46'' \pm 0{\cdot}35'' \\
L &= 6{\cdot}514'' \pm 0{\cdot}016'' \\
\alpha &= 20{\cdot}466'' \pm 0{\cdot}011'' \\
\theta &= 497{\cdot}0^s \pm 1{\cdot}02^s \\
V &= 186\,347 \pm 36 \text{ miles} \\
E &= 0{\cdot}000\,003\,005\,097 \pm 0{\cdot}000\,000\,016\,056
\end{aligned}
\qquad (195)
$$

By substituting these values of the observed quantities in the equations (185), we obtain the following results:

$$1/M = 83{\cdot}7531$$

$$
\begin{aligned}
v_1 &= +0{\cdot}072\,86'' & v_4 &= +0{\cdot}007\,20'' \\
v_2 &= -0{\cdot}005\,32'' & v_5 &= +0{\cdot}029\,98'' \\
v_3 &= -0{\cdot}252\,22'' & v_6 &= +0{\cdot}106\,38''
\end{aligned}
\qquad (196)
$$

Whence, by (193)

$$dE = +0{\cdot}000\,000\,052\,007 \qquad E + dE = 0{\cdot}000\,003\,057\,104 \qquad (197)$$

The value of E given in (195) was obtained from the normal equations (130), and as they must be satisfied with respect to ν and ν^{i}, the value of ν^{ii} deducible from the corrected E of (197) must now be introduced in them, and they must be re-solved. Thus new values will be found for the masses of Mercury and Venus, which in their turn will affect the planetary precession, and through it the entire system of corrections by adjustment. To take account of these changes, some subsidiary formulæ are needed, which will now be investigated.

Reverting to the normal equations (130), if ν^{ii} is regarded as known their general solution will be

$$
\begin{aligned}
100\,\nu &= -3{\cdot}561\,599\,433\,A - 0{\cdot}000\,266\,658\,B \\
100\,\nu^{i} &= -0{\cdot}000\,216\,658\,A - 0{\cdot}013\,762\,852\,B
\end{aligned}
\qquad (198)
$$

where

$$
\begin{aligned}
A &= +\ 19{\cdot}817\,2'' - 21{\cdot}312\,0''\,\nu^{ii} \\
B &= -\ 227{\cdot}771\,8'' + 4\,178{\cdot}625\,2''\,\nu^{ii}
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
\nu &= -0{\cdot}705\,315\,797 + 0{\cdot}749\,994\,745\,\nu^{ii} \\
\nu^{i} &= +0{\cdot}031\,304\,960 - 0{\cdot}575\,051\,828\,\nu^{ii}
\end{aligned}
\qquad (199)
$$

But

$$E + dE = E(1 + \nu^{ii}) \qquad \nu^{ii} = dE/E \tag{200}$$

and from Table III, E = 0·000 002 817 409; whence, by substitution in (199)

$$\begin{aligned} \nu &= -0·705\,315\,80 + 266\,200 dE \\ \nu^{i} &= +0·031\,304\,96 - 204\,107 dE \end{aligned} \tag{201}$$

As these equations are of the form

$$\nu = a + b.dE$$

the equations for correcting the masses of Mercury and Venus will be of the form

$$m + dm = m(1 + a + b.dE) = m(1 + a) + mb.dE \tag{202}$$

where m is the provisional mass of the planet. With

$$m = 0·000\,000\,333\,333 \qquad m^{i} = 0·000\,002\,488\,509$$

from Table III, and the values of $m(1 + a)$ from (132), a comparison of (201) and (202) gives

$$\begin{aligned} \text{Mass of Mercury} &= 0·000\,000\,114\,882 + 0·088\,733 dE \\ \text{Mass of Venus} &= 0·000\,002\,471\,082 - 0·507\,922 dE \end{aligned} \tag{203}$$

Finally, if instead of the values of ν and ν^{i} formerly used, those from (201) are substituted in (76), we shall obtain

$$\mathfrak{L} = 50·358\,6'' - 31\,716'' dE \tag{204}$$

With the value of dE from (197), equation (204) gives

$$\mathfrak{L} = 50·358\,6'' - 0·001\,6'' = 50·357\,0'' \tag{205}$$

which must be used instead of the value given in (195). With it we find

$$v_6 = +0·106\,66'' \tag{206}$$

A repetition of the computation of dE then gives

$$dE = +\,0·000\,000\,052\,000$$

and no further change occurs in $\mathfrak{L}$.

With the residuals from (196) and (206) the formulæ (193) give

$$\begin{aligned} dp &= -0·024\,278'' & dL &= -\ 0·000\,644'' \\ dP &= -0·423\,53'' & d\alpha &= -\ 0·012\,018'' \\ d\mathfrak{L} &= +0·000\,155'' & d\theta &= +\ 0·989\,7^{s} \\ d\mathfrak{A} &= -0·017\,253'' & dV &= -\ 10·855 \text{ miles} \\ dQ &= -0·498\,87'' & dE &= +\ 0·000\,000\,052\,000 \end{aligned} \tag{207}$$

The addition of these corrections to the observed values in (195) and (205), and the substitution of the corrected values in the equations (185), lead to a more exact mass of the Moon and a second series of much diminished residuals, namely

$$1/M = 81\cdot1974$$

$$v_1 = -0\cdot000\,578'' \qquad v_4 = -0\cdot000\,058''$$
$$v_2 = +0\cdot000\,022'' \qquad v_5 = +0\cdot000\,022''$$
$$v_3 = -0\cdot000\,528'' \qquad v_6 = -0\cdot000\,353''$$

To reduce these residuals still further they were substituted in the formulæ (193), and gave the additional corrections

$$\begin{aligned} dp &= +0\cdot000\,043'' & dL &= -0\cdot000\,330'' \\ dP &= -0\cdot000\,038'' & d\alpha &= -0\cdot000\,134'' \\ d\mathfrak{L} &= -0\cdot000\,003'' & d\theta &= +0\cdot001\,238^{s} \\ d\mathfrak{A} &= +0\cdot000\,360'' & dV &= -0\cdot153\,8 \text{ miles} \\ dQ &= +0\cdot000\,920'' & dE &= -0\cdot000\,000\,000\,560 \end{aligned} \qquad (208)$$

When the sums of the respective corrections in (207) and (208) were added to the observed values in (195) and (205), the corrected values so found sufficiently satisfied the equations (185), the largest residual being 0·000 02″, which corresponds to an error of one unit in the sixth decimal place of the logarithms, while the constant logarithms which enter the equations are not trustworthy beyond the fifth decimal place.

It may be well to note that log. M and log. $\mathfrak{A}$ can not be found accurately to six places of decimals unless P is known to 0·000 1″, and certain of the logarithms to at least eight places of decimals. To find log. sin P with the requisite accuracy we have

$$\log.\sin(3\,422'' + x) = 8\cdot219\,834\,949\,4 + 0\cdot000\,126\,900\,8x - 0\cdot000\,000\,018\,5x^2$$

where x is expressed in seconds of arc. The other logarithms necessary to at least eight places are

$$\log.\ 216\,236\cdot648 = 5\cdot334\,929\,300$$
$$\log.\ 3\cdot757\,444\,92 = 0\cdot574\,892\,623$$
$$\log.\ 807\,952\cdot643 = 5\cdot907\,385\,906$$

It is now possible to improve our values of the probable errors of the observed quantities (195), and for that purpose we remark that the expressions (189) are in reality a series of ten observation equations, which determine the four unknowns p, $\mathfrak{L}$, $\mathfrak{A}$ and α from the above-mentioned observed quantities. The corrections by adjustment, dp, dP, $d\mathfrak{L}$, etc., are the residuals of these equations, and the probable error corresponding to weight unity can be computed from them by means of the well-known expression

$$0\cdot674\,49\sqrt{\left(\frac{[pvv]}{m-n}\right)}$$

where $[pvv]$ is the sum of the weighted squares of the residuals, m the number of observation equations, and n the number of unknowns they contain. Accordingly, with the corrections by adjustment from (207) and (208), and the corresponding weights from (189), we find for the probable error of an observation equation of weight unity $\pm 0{\cdot}016\,755$. But by equation (188) we assumed that probable error to be $\pm 0{\cdot}01$, and therefore the probable errors in (195) should all be increased in the ratio of $1{\cdot}675\,5 : 1$.

The probable errors of the adjusted quantities next require consideration, and to avoid ambiguity in the symbol dx, which has hitherto been used to denote sometimes the differential of x, and sometimes the correction by adjustment to the observed value of x, we shall now inclose it in parentheses whenever it has the latter signification. The adjusted value of any observed quantity, x, will then be $x + (dx)$, and the first step towards finding its probable error will be to express $d[x + (dx)]$ in terms of all the observed quantities which it involves. But $d[x + (dx)] = dx + d(dx)$ and the equaions (194) are of the form $(dx) = a(dx) + b(dy) + c(dz) + \text{etc.}$ Further, as the equations (187) were obtained from (185) by reversing the signs of the v's and then differentiating, and as the values of the v's from (187) were substituted in (193) in order to obtain (194), it follows that the latter equations are identical with the general expressions for $-d(dx)$. Accordingly

$$d(dx) = -a.dx - b.dy - c.dz - \text{etc.}$$

and by substituting that value in the expression above

$$d[x + (dx)] = (1 - a)dx - b.dy - c.dz - \text{etc.}$$

Then putting qr_x, qr_y, qr_z, etc., for the probable errors of x, y, z, etc., the probable error of an adjusted quantity, $x + (dx)$, comes out

$$q\sqrt{[(1 - a)^2 r_x^2 + (br_y)^2 + (cr_z)^2 + \text{etc.}]} \tag{209}$$

The coefficients a, b, c, etc., are given in (194); the primitive probable errors r_x, r_y, r_z, etc., are appended to the observed quantities in (195); and q is the ratio r'/r'', r' being the probable error found from the corrections by adjustment for a quantity of weight unity, and r'' the probable error assumed for such a quantity in equation (188), namely 0·01. In Table V the columns headed R_0 contain the primitive probable errors r_x, r_y, r_z, etc., or in other words, the primitive probable errors of p, P, 𝔓, etc.; while the columns headed R_a contain for each of these quantities the numerical value of the coefficient of q in the algorithm (209). Thus the corrected probable error of the observed value of any one of the quantities p, P, 𝔓, etc., will be qR_0, and the probable error of its adjusted value will be qR_a.

TABLE V.—*Constants required for computing the Probable Errors of the Observed and Adjusted Quantities.*

x	R_0	R_a	x	R_0	R_a
	″	″			
p	± 0·008 6	± 0·004 024	a	± 0·011″	± 0·008 987
P	·121	·028 324	θ	$1{\cdot}02^s$	0·218 54
𝔓	·002 48	·002 479	V	36 miles	35·283
𝔑	·011 2	·005 983			
Q	·35	·058 169	E	$\pm \frac{16{\cdot}056}{10^{12}}$	$\pm \frac{4\,203\,8}{10^{13}}$
L	± 0·016	± 0.013 113			

The value of q used in computing the probable errors given in the second and fourth columns of Table VI was that found above, namely, 1·675 5.

From (207) and (208)

$$dE = +0·000\,000\,051\,440$$

which gives, when substituted in (203)

$$\begin{aligned}\text{Mass of Mercury} &= 0·000\,000\,119\,446\\ \text{Mass of Venus} &= 0·000\,002\,444\,954\end{aligned} \qquad (210)$$

By comparing the adjusted mass of the Earth in Table VI with LE VERRIER's value in Table III we find

$$\nu^{ii} = +0·084\,875\,017 \pm 0·002\,499\,980 \qquad (211)$$

and the substitution of that value in (199) gives

$$\begin{aligned}\nu &= -0·641\,659\,980\\ \nu^{i} &= -0·017\,502\,574\end{aligned} \qquad (212)$$

These values of ν, ν^{i}, and ν^{ii} satisfy the first two of the normal equations (130), and when substituted in the weighted observation equations (129) they leave the following residuals:

$+0·093\,1'' \pm 0·023\,1''$	$+0·002\,6'' \pm 0·087\,9''$	$+0·138\,8''$
$-1·048\,6'' \pm 0·167\,9''$	$-0·625\,8'' \pm 0·020\,1''$	$+0·029\,8''$
$-0·527\,1'' \pm 0·026\,0''$	$-0·869\,4'' \pm 0·010\,8''$	$-0·782\,7''$
$-0·610\,4'' \pm 0·003\,3''$	$-0·026\,4'' \pm 0·006\,0''$	$-0·467\,5''$
$+0·245\,9'' \pm 0·000\,0''$	$+0·130\,4'' \pm 0·012\,8''$	$-0·372\,3''$
$-0·224\,1'' \pm 0·010\,8''$	$+0·216\,5'' \pm 0·021\,1''$	$-0·844\,3''$
		$-0·846\,5''$

Taking into account the probable error of ν^{ii}, these residuals give $\pm 0·385\,005''$ for the probable error of any one of the weighted observation equations (129); whence, with the weights from the general solution (198)

$$\begin{aligned}\text{Probable error of } \nu &= \pm 0·072\,659\\ \text{Probable error of } \nu^{i} &= \pm 0·004\,517\end{aligned} \qquad (213)$$

The data employed in this section, the corrections resulting from the least square adjustment, and the corrected values of the quantities investigated, are all brought together in Table VI. In computing the distances of the Sun and Moon from the Earth, the value used for the equatorial radius of the latter body is that given in (3), and the probable errors of the distances have been found from the formula

$$dD = D \cot p . \text{arc } 1''\, dp \qquad (214)$$

where D is the distance of either the Sun or Moon, p the corresponding horizontal parallax, and dp its probable error.

TABLE VI.—*Results for the Epoch* 1850·0, *upon the assumption that the Earth's Flattening is* 1 : 293·47.

Quantities.	Observed values.	Corrections by adjustment.	Adjusted values.
	$''$ $''$	$''$	$''$ $''$
p	8·834 ± 0·014 41	− 0·024 24	8·809 76 ± 0·006 74
P	3 423·08 ± 0·202 74	− 0·423 57	3 422·656 43 ± 0·047 46
$\mathfrak{P}$	50·357 0 ± 0·004 16	+ 0·000 15	50·357 15 ± 0·004 15
$\mathfrak{N}$	9·233 1 ± 0·018 77	− 0·016 89	9·216 21 ± 0·010 02
Q	125·46 ± 0·586 42	− 0·497 95	124·962 05 ± 0·097 46
L	6·514 ± 0·026 81	− 0·000 97	6·513 03 ± 0·021 97
a	20·466 ± 0·018 43	− 0·012 15	20·453 85 ± 0·015 06
θ	497·0^s ± 1·709 01^s	+ 0·990 94^s	497·990 94 ± 0·366 16
V	186 347 ± 60·318 miles	− 11·009 miles	186 335·99 ± 59·117 miles
E	0·000 003 005 097 ± 0·000 000 026 902	+ 51 440	0·000 003 056 537 ± 0·000 000 007 043
M	. . . .	. . . .	0·012 315 7 ± 0·000 042 11

$$\text{Mass of Mercury} = \frac{0·358\,340 \pm 0·072\,659}{3\,000\,000} = \frac{1}{8\,371\,937 \pm 1\,770\,352}$$

$$\text{Mass of Venus} = \frac{0·982\,497 \pm 0·004\,517}{401\,847} = \frac{1}{409\,006 \pm 1\,880}$$

$$\text{Mass of Earth} = \frac{1·084\,875 \pm 0·002\,500}{354\,936} = \frac{1}{327\,168 \pm 754}$$

$$\text{Mass of Moon} = \frac{1}{81·197\,3 \pm 0·277\,6}$$

Mean distance from Earth to Sun $= 92\,793\,500 \pm 70\,993$ miles.
Mean distance from Earth to Moon $= 238\,857 \pm 3·312$ miles.

23.—ADDITIONAL FORMULÆ FOR PRECESSION.

If we put φ'' for the inclination of the moving ecliptic to the fixed ecliptic of 1850·0; θ'' for the longitude of the heliocentric ascending node of the moving ecliptic, counted from the equinox of 1850·0; and t for the time, counted in Julian years from 1850·0; then, according to LE VERRIER*

$$\begin{aligned} p'' &= \varphi'' \sin \theta'' = (g + \Gamma)\, t + r\, t^2 \\ q'' &= \varphi'' \cos \theta'' = (g' + \Gamma')\, t + r' t^2 \end{aligned} \qquad (215)$$

where

$$\begin{aligned} g &= +0·058\,88'' \\ g' &= -0·475\,66'' \\ \Gamma &= +0·006\,27''\,\nu + 0·075\,62''\,\nu^{i} + 0·007\,33''\,\nu^{iii} \\ &\quad -0·024\,96''\,\nu^{iv} - 0·005\,40''\,\nu^{v} + 0·000\,02''\,\nu^{vi} \\ \Gamma' &= -0·005\,25''\,\nu - 0·288\,79''\,\nu^{i} - 0·008\,32''\,\nu^{iii} \\ &\quad -0·160\,09''\,\nu^{iv} - 0·013\,13''\,\nu^{v} - 0·000\,08''\,\nu^{vi} \\ r &= \Sigma \delta_2 p'' \\ r' &= \Sigma \delta_2 q'' \end{aligned} \qquad (216)$$

*41, pp. 49–50, and 8, T. 2, p. 172.

Le Verrier has given only the numerical values of r and r' which correspond to his adopted masses of the planets, and we have now to find the terms involving the variations of these masses. For that purpose we have the equations*

$$
\begin{aligned}
\delta_2 p'' &= -\frac{mn'}{2\mu'} \mathrm{M} t^2 (\xi \cos \tau' - \chi \sin \tau') \\
\delta_2 q'' &= +\frac{mn'}{2\mu'} \mathrm{M} t^2 (\xi \sin \tau' - \chi \cos \tau') \\
\xi &= \tfrac{1}{2}(\delta p - \delta p') \sin \tau' + \tfrac{1}{2}(\delta q - \delta q') \cos \tau' \\
\chi &= \tfrac{1}{2}(\delta p - \delta p') \cos \tau' - \tfrac{1}{2}(\delta q - \delta q') \sin \tau'
\end{aligned}
\tag{217}
$$

whence, by eliminating ξ and χ

$$
\begin{aligned}
\delta_2 p'' &= -\frac{mn'}{4\mu'} \mathrm{M} (\delta q - \delta q') t^2 \\
\delta_2 q'' &= +\frac{mn'}{4\mu'} \mathrm{M} (\delta p - \delta p') t^2
\end{aligned}
\tag{218}
$$

As here written, these equations apply only to the interior planets. To make them general, we put

$$
\mathrm{C}' = \frac{m}{2\mu'} n' \qquad\qquad \mathrm{C} = \frac{m'}{2\mu} a n
$$

and then in (218), instead of $mn'/4\mu'$, the coefficient $\frac{1}{2}\mathrm{C}'$ must be used for planets within the Earth's orbit, and $\frac{1}{2}\mathrm{C}$ for those outside that orbit. By employing Le Verrier's numerical values of C, C′, M, δp, and δq,† we obtain the following expressions:

Action of Mercury.

$$
\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1 + \nu) \left\{ \begin{array}{l} +\ 0\,115\,39'' + 0{\cdot}000\,8''\nu + 0{\cdot}057\,4''\nu^{\mathrm{i}} \\ +\ 0{\cdot}011\,7''\nu^{\mathrm{ii}} + 0{\cdot}001\,6''\nu^{\mathrm{iii}} + 0{\cdot}041\,1''\nu^{\mathrm{iv}} \\ +\ 0{\cdot}002\,6''\nu^{\mathrm{v}} \quad 0{\cdot}000\,0''\nu^{\mathrm{vi}} \end{array} \right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1 + \nu) \left\{ \begin{array}{l} +\ 0{\cdot}095\,17'' + 0{\cdot}001\,0''\nu + 0\,056\,4''\nu^{\mathrm{i}} \\ +\ 0{\cdot}014\,0''\nu^{\mathrm{ii}} + 0{\cdot}001\,5''\nu^{\mathrm{iii}} + 0\,021\,7''\nu^{\mathrm{iv}} \\ +\ 0\,000\,6''\nu^{\mathrm{v}} \quad 0{\cdot}000\,0''\nu^{\mathrm{vi}} \end{array} \right\} t^2
\end{aligned}
\tag{219}
$$

Action of Venus.

$$
\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1 + \nu^{\mathrm{i}}) \left\{ \begin{array}{l} +\ 12{\cdot}636\,47'' - 0{\cdot}019\,9''\nu + 3{\cdot}624\,5''\nu^{\mathrm{i}} \\ +\ 4{\cdot}832\,2''\nu^{\mathrm{ii}} + 0{\cdot}148\,5''\nu^{\mathrm{iii}} + 3{\cdot}839\,6''\nu^{\mathrm{iv}} \\ +\ 0{\cdot}207\,3''\nu^{\mathrm{v}} + 0{\cdot}003\,0''\nu^{\mathrm{vi}} \end{array} \right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1 + \nu^{\mathrm{i}}) \left\{ \begin{array}{l} +\ 2{\cdot}726\,26'' - 0{\cdot}252\,5''\nu + 0{\cdot}949\,1''\nu^{\mathrm{i}} \\ +\ 1{\cdot}265\,6''\nu^{\mathrm{ii}} + 0{\cdot}083\,8''\nu^{\mathrm{iii}} + 0{\cdot}668\,2''\nu^{\mathrm{iv}} \\ +\ 0\,011\,3''\nu^{\mathrm{v}} + 0{\cdot}000\,8''\nu^{\mathrm{vi}} \end{array} \right\} t^2
\end{aligned}
\tag{220}
$$

---

*8, T. 2, pp. 55 and 103.

†8, T. 2, pp. 93–96 and 100–102.

Action of Mars.

$$\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1+\nu^{iii})\left\{\begin{array}{l} +\ 0.383\,62'' + 0.003\,2''\nu + 0.197\,2''\nu^{i} \\ +\ 0.037\,0''\nu^{ii} + 0.006\,0''\nu^{iii} + 0.137\,7''\nu^{iv} \\ +\ 0.002\,3''\nu^{v} + 0.000\,1''\nu^{vi} \end{array}\right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1+\nu^{iii})\left\{\begin{array}{l} +\ 0.361\,57'' + 0.003\,8''\nu + 0.056\,9''\nu^{i} \\ +\ 0.032\,6''\nu^{ii} + 0.005\,3''\nu^{iii} + 0.249\,2''\nu^{iv} \\ +\ 0.013\,5''\nu^{v} + 0.000\,2''\nu^{vi} \end{array}\right\} t^2
\end{aligned} \tag{221}$$

Action of Jupiter.

$$\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1+\nu^{iv})\left\{\begin{array}{l} +\ 5.870\,87'' + 0.089\,3''\nu + 4.939\,1''\nu^{i} \\ +\ 0.003\,6''\nu^{ii} + 0.142\,3''\nu^{iii} + 2.737\,4''\nu^{iv} \\ -\ 2.055\,5''\nu^{v} + 0.014\,5''\nu^{vi} \end{array}\right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1+\nu^{iv})\left\{\begin{array}{l} +\ 2.646\,80'' + 0.106\,5''\nu + 1.292\,5''\nu^{i} \\ -\ 0.000\,5''\nu^{ii} + 0.124\,0''\nu^{iii} - 0.426\,8''\nu^{iv} \\ +\ 1.552\,6''\nu^{v} - 0.009\,6''\nu^{vi} \end{array}\right\} t^2
\end{aligned} \tag{222}$$

Action of Saturn.

$$\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1+\nu^{v})\left\{\begin{array}{l} +\ 0.643\,59'' + 0.004\,1''\nu + 0.228\,1''\nu^{i} \\ \ 0.000\,0''\nu^{ii} + 0.006\,6''\nu^{iii} + 0.386\,9''\nu^{iv} \\ +\ 0.010\,4''\nu^{v} + 0.006\,4''\nu^{vi} \end{array}\right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1+\nu^{v})\left\{\begin{array}{l} -\ 0.145\,62'' + 0.004\,9''\nu + 0.059\,7''\nu^{i} \\ \ 0.000\,0''\nu^{ii} + 0.005\,8''\nu^{iii} - 0.207\,1''\nu^{iv} \\ -\ 0.004\,3''\nu^{v} - 0.004\,8''\nu^{vi} \end{array}\right\} t^2
\end{aligned} \tag{223}$$

Action of Uranus.

$$\begin{aligned}
1\,000\,000\,\delta_2 p'' &= (1+\nu^{vi})\left\{\begin{array}{l} +\ 0.005\,90'' + 0.000\,1''\nu + 0.004\,0''\nu^{i} \\ \ 0.000\,0''\nu^{ii} + 0.000\,1''\nu^{iii} + 0.002\,1''\nu^{iv} \\ -\ 0.000\,4''\nu^{v} \quad 0.000\,0''\nu^{vi} \end{array}\right\} t^2 \\
1\,000\,000\,\delta_2 q'' &= (1+\nu^{vi})\left\{\begin{array}{l} +\ 0.001\,52'' + 0.000\,1''\nu + 0.001\,1''\nu^{i} \\ \ 0.000\,0''\nu^{ii} + 0.000\,1''\nu^{iii} - 0.000\,3''\nu^{iv} \\ +\ 0.000\,3''\nu^{v} \quad 0.000\,0''\nu^{vi} \end{array}\right\} t^2
\end{aligned} \tag{224}$$

The factors given on pages 42 and 72 for converting LE VERRIER's masses into those we have adopted are

$$\begin{array}{lll}
\nu = -0.641\,660 & \nu^{iii} = -0.133\,558 & \nu^{v} = +0.002\,970 \\
\nu^{i} = -0.017\,503 & \nu^{iv} = +0.002\,339 & \nu^{vi} = +0.061\,947 \\
\nu^{ii} = +0.084\,875 & &
\end{array} \tag{225}$$

and by substituting them in the equations (216), and (219) to (224), the following numbers result:

$$g+\Gamma=+0{\cdot}052\,481'' \qquad g'+\Gamma'=-0{\cdot}466\,543''$$

	$\delta_2 p''$	$\delta_2 q''$
Action of Mercury	$+0{\cdot}000\,000\,041\,1''$	$+0{\cdot}000\,000\,033\,9''$
Action of Venus	+ 12 758 7	+ 2 917 6
Action of Mars	+ 329 9	+ 312 6
Action of Jupiter	+ 5 723 0	+ 2 548 2
Action of Saturn	+ 639 3	− 151 8
Action of Uranus	+ 6 1	+ 1 5
Sum	$+0{\cdot}000\,019\,498\,1$	$+0{\cdot}000\,005\,662\,0$

With our values of the masses of the planets we therefore have

$$\begin{aligned}\varphi''\sin\theta''&=+0{\cdot}052\,481''t+0{\cdot}000\,019\,498''t^2\\ \varphi''\cos\theta''&=-0{\cdot}466\,543''t+0{\cdot}000\,005\,662''t^2\end{aligned} \qquad (226)$$

If we restore the terms for taking account of changes in the masses, and put

$$g=+0{\cdot}052\,481'' \qquad \delta=+0{\cdot}000\,019\,498''t+\Gamma$$
$$g'=-0{\cdot}466\,543'' \qquad \delta'=+0{\cdot}000\,005\,662''t+\Gamma'$$

then the equations (226) may be written

$$\varphi''\sin\theta''=gt+\delta t \qquad \varphi''\cos\theta''=g't+\delta' t$$

the solution of which is*

$$\varphi''=(g^2+g'^2)^{\frac{1}{2}}t+\frac{g\delta+g'\delta'}{\sqrt{(g^2+g'^2)}}t$$
$$\tan\theta_0''=g/g'$$
$$\theta''=\theta_0''+\frac{1}{\text{arc }1''}\cdot\frac{g'\delta-g\delta'}{g^2+g'^2}$$

and with the values of Γ and Γ' given in (216), we find from these expressions

$$\begin{aligned}\varphi''&=0{\cdot}469\,486''\,(t-1850)-0{\cdot}000\,003\,447''\,(t-1850)^2\\ &\quad+\left\{\begin{array}{l}+0{\cdot}005\,92''\nu\ \ +0{\cdot}295\,43''\nu^{\text{i}}\\ +0{\cdot}009\,09''\nu^{\text{iii}}+0{\cdot}156\,30''\nu^{\text{iv}}\\ +0{\cdot}012\,44''\nu^{\text{v}}\ +0{\cdot}000\,08''\nu^{\text{vi}}\end{array}\right\}(t-1850)\\ \theta''&=173^\circ\,34'\,54{\cdot}6''\quad-8{\cdot}790\,7''\,(t-1850)\\ &\quad-2\,480''\nu\ \ -18\,832''\nu^{\text{i}}\\ &\quad-2\,792''\nu^{\text{iii}}+18\,760''\nu^{\text{iv}}\\ &\quad-3\,002''\nu^{\text{v}}-\qquad 5''\nu^{\text{vi}}\end{aligned} \qquad (227)$$

To prevent the possibility of misapprehension, it may be well to state explicitly that the νs in the equations (227) relate to the values of the masses adopted in the present paper, and not to LE VERRIER's values.

*S, T. 2, p. 104.

Reverting to the equations (158), and bearing in mind that in them the unit of mass is the Earth without the Moon, we have in the first equation of the group

$$\frac{M}{S} = \frac{EM}{1 + M} \qquad\qquad \frac{r^3}{r_1^3} = \frac{\sin^3 P}{\sin^3 p}$$

whence, by substitution

$$\varepsilon = \frac{EM \sin^3 P}{(1 + M) \sin^3 p} \tag{228}$$

and from the second and third equations

$$\varepsilon = \frac{A\mathfrak{N}}{C\mathfrak{L} - B\mathfrak{N}} \tag{229}$$

$$\varkappa = \frac{C\mathfrak{L} - B\mathfrak{N}}{AC \cos \omega_0} \tag{230}$$

With the adjusted values of p, P, $\mathfrak{L}$, $\mathfrak{N}$, E, and M from Table VI, and the values of A, B, and C from (164), these formulæ give

$$\varepsilon = 2{\cdot}180\,260 \qquad\qquad \varkappa = 17{\cdot}348\,662'' \tag{231}$$

Passing now to the analytical formulæ for precession and nutation given by Serret in his admirable memoir,* after applying the corrections explained in connection with equation (161), putting

$$A = 1 + \frac{3}{2} e_1^2 \qquad\qquad D = \frac{1}{2} - \frac{5}{4} e_1^2$$

$$B = 1 + \frac{3}{2} e_2^2 - 6\gamma^2 \qquad\qquad E = \frac{1}{2} - \frac{5}{4} e_2^2 - \gamma^2$$

$$C = \frac{2\gamma}{\mu} \left(1 + \frac{3}{2} e_2^2 - \frac{5}{2} \gamma^2\right)$$

and making some transformations, we have, in our own notation

$$\left.\begin{aligned}
a &= (A\varkappa + B\varkappa\varepsilon) \cos \omega_0 = \mathfrak{L} \\
b &= \frac{1}{2} (A\varkappa + B\varkappa\varepsilon)\, g' \frac{\cos 2\omega_0}{\sin \omega_0} + \frac{3}{2} \varkappa e_1 e'_1 \cos \omega_0 \\
b &= ag' \cot 2\omega_0 + \frac{3}{2} \varkappa e_1 e'_1 \cos \omega_0 \\
f &= \frac{1}{2} (A\varkappa + B\varkappa\varepsilon)\, g \cos \omega_0 = \frac{1}{2} ag
\end{aligned}\right\} \tag{232}$$

$$\left.\begin{aligned}
P &= a - (g + \Gamma) \cot \omega_0 \\
P' &= b - (ag' + r) \cot \omega_0 + gg' \cot^2 \omega_0 \\
Q &= g' + \Gamma' \\
Q' &= f + r' - ag + \frac{1}{2} g^2 \cot \omega_0 = r' - f + \frac{1}{2} g^2 \cot \omega_0
\end{aligned}\right\} \tag{233}$$

---

*83, pp. 313, 314, 315, and 320.

$$\left.\begin{aligned} \psi &= at + bt^2 + \Psi \\ \omega &= \omega_0 + ft^2 + \Omega \end{aligned}\right\} \qquad (234)$$

$$\left.\begin{aligned} \psi_1 &= \mathrm{P}t + \mathrm{P}'t^2 + \Psi \\ \omega_1 &= \omega_0 + \mathrm{Q}t + \mathrm{Q}'t^2 + \Omega \end{aligned}\right\} \qquad (235)$$

$$\begin{aligned} \Psi = &+ \varkappa\varepsilon\,\mathrm{C}\,\frac{\cos 2\,\omega_0}{\sin \omega_0}\sin ☊ - \varkappa\varepsilon\,\frac{\gamma^2}{\mu}\cos\omega_0 \sin 2\,☊ \\ &- \varkappa\,\frac{\mathrm{D}}{m+a}\cos\omega_0\sin 2\,\odot - \varkappa\varepsilon\,\frac{\mathrm{E}}{m'+a}\cos\omega_0\sin 2\,☾ \\ &+ \varkappa\,\frac{3e_1}{m-\varpi_1}\cos\omega_0\sin \mathrm{A}_\odot + \varkappa\varepsilon\,\frac{3e_2}{m'-\varpi'_1}\cos\omega_0\sin\mathrm{A}_☾ \end{aligned} \qquad (236)$$

$$\begin{aligned} \Omega = &- \varkappa\varepsilon\,\mathrm{C}\cos\omega_0\cos ☊ + \varkappa\varepsilon\frac{\gamma^2}{\mu}\sin\omega_0\cos 2\,☊ \\ &+ \varkappa\,\frac{\mathrm{D}}{m+a}\sin\omega_0\cos 2\,\odot + \varkappa\varepsilon\frac{\mathrm{E}}{m'+a}\sin\omega_0\cos 2\,☾ \end{aligned} \qquad (237)$$

The second and third equations of (158) are respectively identical with the first equation of (232), and with the coefficient of cos ☊ in (237), that coefficient being the quantity known as the constant of nutation. e'_1 is the yearly variation of e_1. In equations (236) and (237) ⊙ and ☾ denote respectively the mean longitudes of the Sun and Moon, while $\mathrm{A}_\odot$ and $\mathrm{A}_☾$ denote the mean anomalies of the same bodies. Usually the symbols ⊙ and ☾ are employed to denote the true longitudes of the Sun and Moon, and on account of the smallness of the terms which they affect, no material error will arise if they are so interpreted in the present case.

As the numerical values of the quantities entering formulæ (232) to (237) are scattered throughout the preceding pages, they are collected here for convenience of reference:

$e_1 = +0{\cdot}016\,771\,049$	$g = +0{\cdot}052\,481''$
$e'_1 = -0{\cdot}000\,000\,424\,5$	$g' = -0{\cdot}466\,543''$
$e_2 = +0{\cdot}054\,899\,720$	$r = +0{\cdot}000\,019\,498''$
$\mu = -0{\cdot}337\,815\,984$	$r' = +0{\cdot}000\,005\,662''$
$\gamma = +0{\cdot}044\,886\,793$	$\omega_0 = 23^\circ\,27'\,31{\cdot}36''$

$$m = \frac{2\pi.365\tfrac{1}{4}}{365{\cdot}256\,357\,8} = 6{\cdot}283\,075\,940$$

$$m' = \frac{2\pi.365\tfrac{1}{4}}{27{\cdot}321\,661\,16} = 83{\cdot}996\,848\,52$$

$\varpi_1 =$ the sidereal motion of the solar perigee in $365\tfrac{1}{4}$ days $= 11{\cdot}3618'' + 0{\cdot}1375''$ $= 11{\cdot}4993''$ according to HANSEN*; or $61{\cdot}6995'' - 50{\cdot}2357'' = 11{\cdot}4638''$ according to LE VERRIER†. We adopt the mean, namely, $+11{\cdot}4816''$ $= 0{\cdot}000\,055\,664$ of radius.

$\varpi'_1 =$ the sidereal motion of the lunar perigee in $365\tfrac{1}{4}$ days, which is according to HANSEN‡ $\frac{d}{dt}\left(\omega - \Theta - \psi_1\right) = 216\,115{\cdot}2207'' - 69\,629{\cdot}3961'' - 50{\cdot}2230''$ $= +146\,435{\cdot}6016'' = 0{\cdot}709\,939\,830$ of radius.

*54, p, 16. †41, pp. 102 and 51. ‡54, pp. 15 and 16.

In (236) and (237) ω_0 is the true obliquity of the ecliptic at the instant for which Ψ and Ω are required, and we should there take

$$\omega_0 = 23^\circ\, 27'\, 31{\cdot}36'' - 0{\cdot}466\,54''\,(t - 1850) - 0{\cdot}000\,000\,73''\,(t - 1850)^2$$

whence

$$\begin{aligned} \text{Log.}\, \frac{\cos 2\,\omega_0}{\sin \omega_0} &= 0{\cdot}234\,4742 + 0{\cdot}000\,004\,365\,(t - 1850) \\ \text{Log.}\ \cos \omega_0 &= 9{\cdot}962\,5337 + 0{\cdot}000\,000\,426\,(t - 1850) \\ \text{Log.}\ \sin \omega_0 &= 9{\cdot}599\,9792 - 0{\cdot}000\,002\,264\,(t - 1850) \end{aligned}$$

These numbers must now be substituted in the formulæ under consideration, and we shall follow LE VERRIER and SERRET in first giving the results for luni-solar precession, and for nutation, with $\mathfrak{L}$, $\varkappa$ and ε retained as symbols. They are

$$\begin{aligned} a &= +[9{\cdot}962\,7169 - 10]\,\varkappa + [9{\cdot}959\,2345 - 10]\,\varkappa\varepsilon = \mathfrak{L} \\ b &= -[4{\cdot}290\,2802 - 10]\,\varkappa - [4{\cdot}284\,6117 - 10]\,\varkappa\varepsilon \\ b &= -[4{\cdot}325\,3772 - 10]\,\mathfrak{L} - [1{\cdot}991\,06 - 10]\,\varkappa \\ f &= +[3{\cdot}067\,264 - 10]\,\varkappa + [3{\cdot}063\,782 - 10]\,\varkappa\varepsilon \\ f &= +[3{\cdot}104\,547 - 10]\,\mathfrak{L} \end{aligned} \qquad (238)$$

$$\begin{aligned} \Psi = &-[9{\cdot}658\,7186 + 0{\cdot}000\,004\,365\,(t - 1850) - 10]\,\varkappa\varepsilon \sin ☊ \\ &+[7{\cdot}738\,09 \quad + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa\varepsilon \sin 2\,☊ \\ &-[8{\cdot}863\,010 \quad + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa \sin 2\,\odot \\ &-[7{\cdot}732\,19 \quad + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa\varepsilon \sin 2\,☾ \\ &+[7{\cdot}866\,05 \quad + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa \sin \mathrm{A}_\odot \\ &+[7{\cdot}258\,65 \quad + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa\varepsilon \sin \mathrm{A}_☾ \end{aligned} \qquad (239)$$

$$\begin{aligned} \Omega = &+[9{\cdot}386\,7779 + 0{\cdot}000\,000\,426\,(t - 1850) - 10]\,\varkappa\varepsilon \cos ☊ \\ &-[7{\cdot}375\,54 \quad - 0{\cdot}000\,002\,264\,(t - 1850) - 10]\,\varkappa\varepsilon \cos 2\,☊ \\ &+[8{\cdot}500\,455 \quad - 0{\cdot}000\,002\,264\,(t - 1850) - 10]\,\varkappa \cos 2\,\odot \\ &+[7{\cdot}369\,63 \quad - 0{\cdot}000\,002\,264\,(t - 1850) - 10]\,\varkappa\varepsilon \cos 2\,☾ \end{aligned} \qquad (240)$$

After substituting the adjusted value of $\mathfrak{L}$ from Table VI, and the values of $\varkappa$ and ε from (231), the formulæ (238) to (240), in connection with (233) to (235), give

$$\begin{aligned} \psi &= \left\{ \begin{matrix} +50{\cdot}357\,15'' + 0{\cdot}0144''\,\nu \ + 0{\cdot}1743''\,\nu^{\mathrm{i}} \\ +0{\cdot}0169''\,\nu^{\mathrm{iii}} - 0{\cdot}0575''\,\nu^{\mathrm{iv}} - 0{\cdot}0124''\,\nu^{\mathrm{v}} \end{matrix} \right\} (t - 1850) \\ &\quad - 0{\cdot}000\,106\,69''\,(t - 1850)^2 + \Psi \\ \omega &= 23^\circ\, 27'\, 31{\cdot}47'' + 0{\cdot}000\,006\,41''\,(t - 1850)^2 + \Omega \\ \psi_1 &= 50{\cdot}236\,22''\,(t - 1850) + 0{\cdot}000\,110\,22''\,(t - 1850)^2 + \Psi \\ \omega_1 &= 23^\circ\, 27'\, 31{\cdot}47'' - \left\{ \begin{matrix} +0{\cdot}466\,54'' \ + 0{\cdot}0053''\,\nu \ + 0{\cdot}2888''\,\nu^{\mathrm{i}} \\ +0{\cdot}0083''\,\nu^{\mathrm{iii}} + 0{\cdot}1601''\,\nu^{\mathrm{iv}} + 0{\cdot}0131''\,\nu^{\mathrm{v}} \end{matrix} \right\} (t - 1850) \\ &\quad - 0{\cdot}000\,000\,73''\,(t - 1850)^2 + \Omega \end{aligned} \qquad (241)$$

$$
\begin{aligned}
\Psi = & -\{17.2382'' + 0.000\,173\,2''(t-1850)\} \sin \Omega \\
& +\{\ 0.2070'' + 0.000\,000\,2''\,(t-1850)\} \sin 2\,\Omega \\
& -\{\ 1.2655'' + 0.000\,001\,2''\,(t-1850)\} \sin 2\,\odot \\
& -\{\ 0.2042'' + 0.000\,000\,2''\,(t-1850)\} \sin 2\,☾ \\
& +\{\ 0.1274'' + 0.000\,000\,1''\,(t-1850)\} \sin A_{\odot} \\
& +\{\ 0.0686'' + 0.000\,000\,1''\,(t-1850)\} \sin A_{☾}
\end{aligned}
\tag{242}
$$

$$
\begin{aligned}
\Omega = & +\{9.216\,2'' + 0.000\,009\,0''\,(t-1850)\} \cos \Omega \\
& -\{0.089\,8'' - 0.000\,000\,5''\,(t-1850)\} \cos 2\Omega \\
& +\{0.549\,2'' - 0.000\,002\,9''\,(t-1850)\} \cos 2\odot \\
& +\{0.088\,6'' - 0.000\,000\,5''\,(t-1850)\} \cos 2☾
\end{aligned}
\tag{243}
$$

Respecting ω_0, it is to be remarked that LE VERRIER'S data* give $\omega_1 = 23^\circ\ 27'\ 49.804''$ for the epoch 1810·7, and by bringing that up to 1850·0 with our yearly motion, $-0.466\,543''$, the value of ω_0 given in (241) is obtained. Of course the νs in (241) relate to the masses of the planets adopted in this paper, and not to LE VERRIER'S masses.

It yet remains to deduce from the group of formulæ (241) the ten quantities usually employed in computations relating to precession, and these quantities will be given in a shape permitting of their ready reference to any desired equinox and ecliptic. The transformation from one equinox and ecliptic to another might be effected by HANSEN'S formulæ,† but they involve the constants g, g', r and r', whose values are not always known, and to avoid that difficulty we shall develop formulæ involving only the coefficients in equations (234) and (235).

If $d\psi/dt$ is the annual change of ψ at the instant t_0, then by (232) and (234)

$$\frac{d\psi}{dt} = a\,\frac{\cos\omega_1}{\cos\omega_0}$$

$$\psi = a\,\frac{\cos\omega_1}{\cos\omega_0}(t-t_0) + b(t-t_0)^2 \tag{244}$$

But by (235), which relates to the epoch T,

$$\omega_1 = \omega_0 + Q(t_0 - T) + Q'(t_0 - T)^2 \tag{245}$$

whence, with sufficient accuracy

$$\cos\omega_1 = \cos\omega_0 - Q\sin\omega_0(t_0 - T)$$

and by substituting that value in (244), the expression for ψ given in (273) results. Further, as in the second equation of (234) ω_0 is the true obliquity of the ecliptic at the instant T, in order to change the epoch from T to t_0 we have only to substitute ω_1 for ω_0; and by so doing the expression for ω given in (273) was obtained.

*41, p. 51. †78, cols. 114, 139, and 154.

The expressions for ψ_1 and ω_1, in (235), are of the form

$$u = A + B(t - T) + C(t - T)^2$$

where

$$B = -\frac{du}{dT} \qquad C = \tfrac{1}{2}\frac{d^2u}{dT^2}$$

and consequently, the values which they assume when the origin of time is changed from T to t_0 may be found by the algorithm

$$\begin{aligned} u' = {} & A + B(t_0 - T) + C(t_0 - T)^2 \\ & + [B + 2\,C(t_0 - T)]\,(t - t_0) + C(t - t_0)^2 \end{aligned}$$

The results are given in (274).

If λ is the planetary precession during the interval $(t - t_0)$, and $\tfrac{1}{2}(\omega_1 + \omega) = \omega_0 + d\omega_0$, then*

$$\begin{aligned} \lambda &= (\psi - \psi_1) \sec(\omega_0 + d\omega_0) \\ &= (\psi - \psi_1) \sec \omega_0 + (\psi - \psi_1) d\omega_0 \sec \omega_0 \tan \omega_0 \end{aligned} \qquad (246)$$

But from (273) and (274)

$$\begin{aligned} \psi - \psi_1 &= [a - P - (2P' + aQ \tan \omega_0)(t_0 - T)](t - t_0) + (b - P')(t - t_0)^2 \\ \tfrac{1}{2}(\omega_1 + \omega) &= \omega_0 + Q(t_0 - T) + Q'(t_0 - T)^2 \\ &\quad + [\tfrac{1}{2} Q + Q'(t_0 - T)](t - t_0) + \tfrac{1}{2}(f + Q')(t - t_0)^2 \end{aligned} \qquad (247)$$

whence

$$d\omega_0 = Q(t_0 - T) + Q'(t_0 - T)^2 + [\tfrac{1}{2}Q + Q'(t_0 - T)](t - t_0) + \tfrac{1}{2}(f + Q')(t - t_0)^2 \qquad (248)$$

and by substituting these values of $(\psi - \psi_1)$ and $d\omega_0$ in (246), and rejecting all terms above the second order with respect to $(t_0 - T)$ and $(t - t_0)$, the expression for λ given in (273) results.

To find m and n we have the well-known formulæ

$$m = \frac{d\psi}{dt} \cos \omega - \frac{d\lambda}{dt} \qquad n = \frac{d\psi}{dt} \sin \omega \qquad (249)$$

But from (273)

$$\begin{aligned} \frac{d\psi}{dt} &= a - aQ \tan \omega_0 (t_0 - T) + 2b(t - t_0) \\ \frac{d\lambda}{dt} &= + \sec \omega_0 (a - P) - \sec \omega_0 (2P' + PQ \tan \omega_0)(t_0 - T) \\ &\quad + 2 \sec \omega_0 [b - P' + \tfrac{1}{2}Q(a - P) \tan \omega_0](t - t_0) \\ \omega &= \omega_0 + Q(t_0 - T) + Q'(t_0 - T)^2 + f(t - t_0)^2 \\ \sin \omega &= \sin \omega_0 + [Q(t_0 - T) + Q'(t_0 - T)^2 + f(t - t_0)^2] \cos \omega_0 \\ \cos \omega &= \cos \omega_0 - [Q(t_0 - T) + Q'(t_0 - T)^2 + f(t - t_0)^2] \sin \omega_0 \end{aligned}$$

* 2, vol. 1, p. 607.

and by substituting these values in the formulæ (249), and rejecting all terms above the second order with respect to $(t_0 - \mathrm{T})$ and $(t - t_0)$, we find

$$\begin{aligned} m = &+ a \cos \varpi_0 - (a - \mathrm{P}) \sec \varpi_0 \\ &+ [(2\mathrm{P}' + \mathrm{PQ} \tan \varpi_0) \sec \varpi_0 - 2a\mathrm{Q} \sin \varpi_0](t_0 - \mathrm{T}) \\ &+ a \sin \varpi_0(\mathrm{Q}^2 \tan \varpi_0 - \mathrm{Q}')(t_0 - \mathrm{T})^2 \\ &- \left\{ \begin{array}{l} + [2(b - \mathrm{P}') + \mathrm{Q}(a - \mathrm{P}) \tan \varpi_0] \sec \varpi_0 \\ - 2b \cos \varpi_0 + 2b\mathrm{Q} \sin \varpi_0(t_0 - \mathrm{T}) \end{array} \right\} (t - t_0) \\ &- af \sin \varpi_0(t - t_0)^2 \end{aligned} \tag{250}$$

$$\begin{aligned} n = &+ a \sin \varpi_0 + a\mathrm{Q}(\cos \varpi_0 - \sin \varpi_0 \tan \varpi_0)(t_0 - \mathrm{T}) \\ &- a \cos \varpi_0(\mathrm{Q}^2 \tan \varpi_0 - \mathrm{Q}')(t_0 - \mathrm{T})^2 \\ &+ 2b[\sin \varpi_0 + \mathrm{Q} \cos \varpi_0(t_0 - \mathrm{T})](t - t_0) \\ &+ af \cos \varpi_0(t - t_0)^2 \end{aligned} \tag{251}$$

When converted into numbers by means of the coefficients in (241), these expressions become

$$\begin{aligned} m = &+ 46{\cdot}063\,15'' \\ &+ 0{\cdot}000\,277\,23''\,(t_0 - 1850) + 0{\cdot}000\,000\,000\,115''\,(t_0 - 1850)^2 \\ &+ [0{\cdot}000\,277\,29'' - 0{\cdot}000\,000\,000\,192''\,(t_0 - 1850)](t - t_0) \\ &- 0{\cdot}000\,000\,000\,623''\,(t - t_0)^2 \end{aligned} \tag{252}$$

$$\begin{aligned} n = &+ 20{\cdot}046\,61'' \\ &- 0{\cdot}000\,084\,81''\,(t_0 - 1850) - 0{\cdot}000\,000\,000\,266''\,(t_0 - 1850)^2 \\ &- [0{\cdot}000\,084\,94'' - 0{\cdot}000\,000\,000\,443''\,(t_0 - 1850)](t - t_0) \\ &+ 0{\cdot}000\,000\,001\,435''\,(t - t_0)^2 \end{aligned} \tag{253}$$

It is, therefore, evident that when neither $(t_0 - \mathrm{T})$ nor $(t - t_0)$ exceeds a century, all second order terms may be neglected, and the algebraic expressions for m and n may be written as in (273).

Expressions for φ'' and θ'' have already been found from g, g', r and r', but we have now to deduce them from the coefficients in (234) and (235). For that purpose we shall employ the equations*

$$\varphi''^2 = (\varpi_1 - \varpi)^2 + \lambda^2 \sin^2 \tfrac{1}{2}(\varpi_1 + \varpi) \tag{254}$$

$$\tan\,(\theta'' + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = \frac{\lambda}{\varpi_1 - \varpi} \sin \tfrac{1}{2}(\varpi_1 + \varpi) \tag{255}$$

The first of these equations may be written

$$\varphi'' = -(\varpi_1 - \varpi) \left\{ 1 + \frac{\lambda^2 \sin^2 \frac{1}{2}(\varpi_1 + \varpi)}{(\varpi_1 - \varpi)^2} \right\}^{\frac{1}{2}}$$

whence, expanding by the binomial theorem

$$\varphi'' = -(\varpi_1 - \varpi) - \frac{\lambda^2 \sin^2 \frac{1}{2}(\varpi_1 + \varpi)}{2(\varpi_1 - \varpi)} + \text{etc.} \tag{256}$$

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* 2, vol. 1, p. 607.

If now we put

$$\begin{aligned} \tfrac{1}{2}(\varpi_1 + \varpi) &= \varpi_0 + d\varpi_0 \\ A &= a - P - (2P' + PQ \tan \varpi_0)(t_0 - T) \\ B &= b - P' + \tfrac{1}{2}Q(a - P) \tan \varpi_0 \end{aligned} \qquad (257)$$

we shall have from (248), (273), and (274)

$$\begin{aligned} d\varpi_0 &= Q(t_0 - T) + \tfrac{1}{2}Q(t - t_0) + \text{etc.} \\ \lambda &\doteq \sec \varpi_0 A(t - t_0) + \sec \varpi_0 B(t - t_0)^2 \end{aligned} \qquad (258)$$

$$\varpi_1 - \varpi = Q(t - t_0) + 2Q'(t_0 - T)(t - t_0) - (f - Q')(t - t_0)^2 \qquad (259)$$

Whence, with sufficient accuracy

$$\begin{aligned} \sin \tfrac{1}{2}(\varpi_1 + \varpi) &= \sin \varpi_0 + d\varpi_0 \cos \varpi_0 \\ \lambda^2 \sin^2 \tfrac{1}{2}(\varpi_1 + \varpi) &= + \tan^2 \varpi_0[A^2(t + t_0)^2 + 2AB(t - t_0)^3] \\ &\quad + 2A^2 d\varpi_0 \tan \varpi_0(t - t_0)^2 \end{aligned} \qquad (260)$$

The substitution of (259) and (260) in (256) gives, after rejecting all terms above the second order with respect to $(t_0 - T)$ and $(t - t_0)$

$$\begin{aligned} = &- [Q + \tan^2 \varpi_0(a - P)^2/2Q](t - t_0) \\ &- \left\{ \begin{array}{l} + 2Q' + \tan \varpi_0(a - P)^2 \\ - \tan^2 \varpi_0(a - P)(2P' + PQ \tan \varpi_0)/Q \end{array} \right\} (t_0 - T)(t - t_0) \\ &+ \{(f - Q') - \tan^2 \varpi_0(a - P)[b - P' + \tfrac{1}{2}Q(a - P) \tan \varpi_0]/Q\}(t - t_0)^2 \end{aligned} \qquad (261)$$

and as the quantities involved are so related that the coefficient of $(t_0 - T)$ is known to be exactly twice that of $(t - t_0)^2$, (261) naturally takes the form given in (274).

Reverting now to (255); after eliminating λ by means of the relation *

$$\psi - \psi_1 = \lambda \cos \tfrac{1}{2}(\varpi_1 + \varpi)$$

and putting

$$\tan (\theta'' + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = \tan (\theta_0'' + d\theta_0'') = \tan \theta_0'' + d\theta_0'' \sec^2 \theta_0'' \qquad (262)$$

we have

$$\tan \theta_0'' + d\theta_0'' \sec^2 \theta_0'' = \frac{\psi - \psi_1}{\varpi_1 - \varpi} \tan \tfrac{1}{2}(\varpi_1 + \varpi) \qquad (263)$$

But, from (247) and (259)

$$\frac{\psi - \psi_1}{\varpi_1 - \varpi} = \frac{a - P - (2P' + aQ \tan \varpi_0)(t_0 - T) + (b - P')(t - t_0)}{Q\left\{1 + \frac{2Q'}{Q}(t_0 - T) - \frac{f - Q'}{Q}(t - t_0)\right\}}$$

Whence, with sufficient accuracy, through an expansion by the binomial theorem

$$\begin{aligned} \frac{\psi - \psi_1}{\varpi_1 - \varpi} = \frac{a - P}{Q} &- \left\{ \frac{2P' + aQ \tan \varpi_0}{Q} + \frac{2Q'(a - P)}{Q^2} \right\} (t_0 - T) \\ &+ \left\{ \frac{b - P'}{Q} + \frac{(a - P)(f - Q')}{Q^2} \right\} (t - t_0) \end{aligned} \qquad (264)$$

* 2, vol. 1, p. 607.

Also, by putting $\frac{1}{2}(\varpi_1+\varpi)=\varpi_0+d\varpi_0$, and taking the value of $d\varpi_0$ from (258)

$$\begin{aligned}\tan\tfrac{1}{2}(\varpi_1+\varpi)&=\tan\varpi_0+d\varpi_0\sec^2\varpi_0\\&=\tan\varpi_0+\sec^2\varpi_0[\mathrm{Q}\,(t_0-\mathrm{T})+\tfrac{1}{2}\mathrm{Q}(t-t_0)]\end{aligned}\tag{265}$$

A comparison of (263) with the result obtained by multiplying together (264) and (265) shows that, to terms of the first order with respect to $(t_0-\mathrm{T})$ and $(t-t_0)$

$$\tan\theta_0''=\tan\varpi_0(a-\mathrm{P})/\mathrm{Q}\tag{266}$$

$$\begin{aligned}d\theta_0''\sec^2\theta_0''=&-\tan\varpi_0[(2\mathrm{P}'+a\mathrm{Q}\tan\varpi_0)/\mathrm{Q}+2\mathrm{Q}'(a-\mathrm{P})/\mathrm{Q}^2](t_0-\mathrm{T})\\&+(a-\mathrm{P})\sec^2\varpi_0(t_0-\mathrm{T})\\&+\tan\varpi_0[(b-\mathrm{P}')/\mathrm{Q}+(a-\mathrm{P})(f-\mathrm{Q}')/\mathrm{Q}^2](t-t_0)\\&+\tfrac{1}{2}(a-\mathrm{P})\sec^2\varpi_0(t-t_0)\end{aligned}\tag{267}$$

As (262) gives

$$\theta''=\theta_0''+d\theta_0''-\tfrac{1}{2}(\psi+\psi_1)$$

by taking the values of θ_0'' and $d\theta_0''$ from (266) and (267), and remembering that to terms of the first order

$$\tfrac{1}{2}(\psi+\psi_1)=\tfrac{1}{2}(a+\mathrm{P})(t-t_0)$$

we find

$$\begin{aligned}\theta''=\theta_0''-&\frac{\tan\varpi_0\cos^2\theta_0''}{\mathrm{Q}}[2\mathrm{P}'+a\mathrm{Q}\tan\varpi_0+2\mathrm{Q}'(a-\mathrm{P})/\mathrm{Q}](t_0-\mathrm{T})\\&+(a-\mathrm{P})\sec^2\varpi_0\cos^2\theta_0''(t_0-\mathrm{T})\\&+\frac{\tan\varpi_0\cos^2\theta_0''}{\mathrm{Q}}[b-\mathrm{P}'+(a-\mathrm{P})(f-\mathrm{Q}')/\mathrm{Q}](t-t_0)\\&+\tfrac{1}{2}[(a-\mathrm{P})\sec^2\varpi_0\cos^2\theta_0''-(a+\mathrm{P})](t-t_0)\end{aligned}\tag{268}$$

If we had not transformed (255), but had developed

$$\frac{\lambda}{\varpi_1-\varpi}\sin\tfrac{1}{2}(\varpi_1+\varpi)$$

instead of

$$\frac{\psi-\psi_1}{\varpi_1-\varpi}\tan\tfrac{1}{2}(\varpi_1+\varpi)$$

we would have obtained the slightly more complicated expression

$$\begin{aligned}\theta''=\theta_0''-&\frac{\tan\varpi_0\cos^2\theta_0''}{\mathrm{Q}}[2\mathrm{P}'+\mathrm{PQ}\tan\varpi_0+2\mathrm{Q}'(a-\mathrm{P})/\mathrm{Q}](t_0-\mathrm{T})\\&+\cos^2\theta_0''(a-\mathrm{P})(t_0-\mathrm{T})\\&+\frac{\tan\varpi_0\cos^2\theta_0''}{\mathrm{Q}}[b-\mathrm{P}'+\tfrac{1}{2}\mathrm{Q}(a-\mathrm{P})\tan\varpi_0+(a-\mathrm{P})(f-\mathrm{Q}')/\mathrm{Q}](t-t_0)\\&+\tfrac{1}{2}[(a-\mathrm{P})\cos^2\theta_0''-(a+\mathrm{P})](t-t_0)\end{aligned}\tag{269}$$

If M is the longitude of the ascending node of the mean ecliptic at the time $t + dt$, reckoned from the equinox of date t, upon the ecliptic of date t, then

$$M = \theta'' + \frac{d\theta''}{dt} t + \psi_1$$

But

$$\theta'' = \theta_0'' + \frac{d\theta_0''}{dt} t \qquad (270)$$

and therefore, with sufficient accuracy

$$M = \theta_0'' + 2\frac{d\theta_0''}{dt} t + \psi_1 \qquad (271)$$

As the relation of M to θ'' is the same as that of ω_1 to ω, when the date of the equinox and ecliptic is changed from T to t_0 in (270) and (271), we must evidently have

$$\begin{aligned} \theta'' &= \theta_0'' + \frac{d\theta_0''}{dt}[2(t_0 - T) + (t - t_0)] + \frac{d\psi_1}{dt}(t_0 - T) \\ M &= \theta_0'' + \left\{ 2\frac{d\theta_0''}{dt} + \frac{d\psi_1}{dt} \right\} [(t_0 - T) + (t - t_0)] \end{aligned} \qquad (272)$$

and these are the forms adopted in (274); the values of θ_0'' and $d\theta_0''/dt$ being taken from (266) and (268).

$$\begin{aligned} \psi &= a[1 - Q \tan \omega_0(t_0 - T)](t - t_0) + b(t - t_0)^2 + \Psi \\ \omega &= \omega_0 + Q(t_0 - T) + Q'(t_0 - T)^2 + f(t - t_0)^2 + \Omega \\ \lambda &= + \sec \omega_0[a - P - (2P' + PQ \tan \omega_0)(t_0 - T)](t - t_0) \\ &\quad + \sec \omega_0[b - P' + \tfrac{1}{2}Q(a - P) \tan \omega_0](t - t_0)^2 \\ m &= + a \cos \omega_0 - (a - P) \sec \omega_0 \\ &\quad + \{(2P' + PQ \tan \omega_0) \sec \omega_0 - 2aQ \sin \omega_0\} \{(t_0 - T) + (t - t_0)\} \\ n &= a \sin \omega_0 + 2b \sin \omega_0[(t_0 - T) + (t - t_0)] \end{aligned} \qquad (273)$$

$$\begin{aligned} \psi_1 &= [P + 2P'(t_0 - T)](t - t_0) + P'(t - t_0)^2 + \Psi \\ \omega_1 &= \omega_0 + Q(t_0 - T) + Q'(t_0 - T)^2 \\ &\quad + [Q + 2Q'(t_0 - T)](t - t_0) + Q'(t - t_0)^2 + \Omega \\ \varphi'' &= - [Q + \tan^2 \omega_0(a - P)^2/2Q](t - t_0) \\ &\quad + \{(f - Q') - \tan^2 \omega_0(a - P)[b - P' + \tfrac{1}{2}Q(a - P) \tan \omega_0]/Q\}\{2(t_0 - T) \\ &\qquad (t - t_0) + (t - t_0)^2\} \\ \theta_0'' &= \text{arc tan} [\tan \omega_0(a - P)/Q] \\ \Lambda &= + \frac{\tan \omega_0 \cos^2 \theta_0''}{Q}[b - P' + (a - P)(f - Q')/Q] \\ &\quad + \tfrac{1}{2}(a - P) \sec^2 \omega_0 \cos^2 \theta_0'' - \tfrac{1}{2}(a + P) \\ \theta'' &= \theta_0'' + \Lambda [2(t_0 - T) - (t - t_0)] + P(t_0 - T) \\ M &= \theta_0'' + (2\Lambda + P)[(t_0 - T) + (t - t_0)] \end{aligned} \qquad (274)$$

By means of the groups of formulæ (273) and (274) the entire system of quantities used in computing precession can be readily found for any desired equinox and ecliptic when the expressions for ψ, ψ_1, ω, and ω_1 are known for a given equinox and ecliptic. With respect to the notation, it may be well to remark that ψ, ω, ω_1, φ'', and θ'' are the quantities which Dr. PETERS designated ψ', θ', θ_1, π'', and Π. Further, T is the epoch for which the constants in the formulæ (234) and (235) were originally computed, while t_0 is the date of the new equinox and ecliptic to which the precession is to be referred, and t is the date for which the various quantities are required. In (273) and (274) all the angles and circular functions are expressed in parts of radius, and to convert them into seconds of arc it yet remains to introduce the factor arc 1″ wherever necessary.

By substituting the coefficients from (241) in the formulæ (273) and (274), all the following numerical expressions were obtained, except those for φ'', θ'', and M, which are from (227).

$$
\begin{aligned}
\psi &= [50{\cdot}357\,15'' + 0{\cdot}000\,049\,43''\,(t_0 - 1850)]\,(t - t_0) \\
&\qquad - 0{\cdot}000\,106\,69''\,(t - t_0)^2 + \Psi \\
\omega &= 23^\circ\,27'\,31{\cdot}47'' - 0{\cdot}466\,54''(t_0 - 1850) \\
&\quad - 0{\cdot}000\,000\,73''\,(t_0 - 1850)^2 + 0{\cdot}000\,006\,41''\,(t - t_0)^2 + \Omega \\
\lambda &= [0{\cdot}131\,83'' - 0{\cdot}000\,186\,55''\,(t_0 - 1850)](t - t_0) \\
&\qquad - 0{\cdot}000\,236\,52''\,(t - t_0)^2 \\
m &= 46{\cdot}063\,15'' + 0{\cdot}000\,277\,23''\,[(t_0 - 1850) + (t - t_0)] \\
n &= 20{\cdot}046\,61'' - 0{\cdot}000\,084\,94''\,[(t_0 - 1850) + (t - t_0)]
\end{aligned} \tag{275}
$$

$$
\begin{aligned}
\psi_1 &= [50{\cdot}236\,22'' + 0{\cdot}000\,220\,44''\,(t_0 - 1850)](t - t_0) \\
&\qquad + 0{\cdot}000\,110\,22''\,(t - t_0)^2 + \Psi \\
\omega_1 &= 23^\circ\,27'\,31{\cdot}47'' - 0{\cdot}466\,54''\,(t_0 - 1850) - 0{\cdot}000\,000\,73''\,(t_0 - 1850)^2 \\
&\quad - [0{\cdot}466\,54'' + 0{\cdot}000\,001\,46''\,(t_0 - 1850)](t - t_0) \\
&\quad - 0{\cdot}000\,000\,73''\,(t - t_0)^2 + \Omega \\
\varphi'' &= [0{\cdot}469\,49'' - 0{\cdot}000\,006\,89''\,(t_0 - 1850)](t - t_0) \\
&\qquad - 0{\cdot}000\,003\,45''\,(t - t_0)^2 \\
\theta'' &= 173^\circ\,34'\,55'' + 32{\cdot}655''\,(t_0 - 1850) - 8{\cdot}791''\,(t - t_0) \\
\mathrm{M} &= 173^\circ\,34'\,55'' + 32{\cdot}655''\,[(t_0 - 1850) + (t - t_0)]
\end{aligned} \tag{276}
$$

In Section 25 we shall see that the outcome of the present investigation respecting precession depends somewhat upon the value adopted for the flattening of the Earth, but formulæ (227), (241), (275), and (276) are very nearly correct, and it is desirable that ready means should be provided for comparing them with the results obtained by other astronomers. To that end the coefficients of the formulæ of LA PLACE,* BESSEL,† STRUVE and PETERS,‡ HANSEN,§ LE VERRIER,‖ and the present writer, are collected in

*7, T. 3, liv. 6, chap. 16, converted into sexagessimal seconds by BESSEL, 1, p. 285, and 1½, p. iv.

†1½, p. v.

‡109, pp. 190 and 195.

§78, cols. 113, 114, 139, and 152–154.

‖8, T. 2, p. 174, and T. 4, p. 104.

Table VII, where they are all referred to the equinox and ecliptic of 1800; such of the formulæ as were referred by their authors to equinoxes and ecliptics other than those of 1800 having been reduced to that epoch by means of the formulæ (273) and (274), without the introduction of any extraneous factors. Respecting HANSEN's formulæ, it should be remarked that the values of ω_0 and P given in Table VII differ slightly from those in his Tables du Soleil, p. 5; but the P agrees with that in his Tables de la Lune, p. 16. The dates of publication are, for the article in the Astronomische Nachrichten, September, 1852; for the solar tables, 1853; and for the lunar tables, 1857.

TABLE VII.—*Values given by various Authors for the Coefficients in Formulæ* (234) *and* (235); *said Values being all referred to the Equinox and Ecliptic of* 1800.

Author.	LA PLACE.	BESSEL.	STRUVE and PETERS.	HANSEN.	LE VERRIER.	Formulæ (241).
Date.	1802.	1826.	1841.	1852.	1856.	1889.
a	+ 50·290 34″	+ 50·378 26″	+ 50·379 8 ″	+ 50·355 93″	+ 50·368 88″	+ 50·354 68″
1 000 b	− 0·121 79	− 0·121 79	− 0·108 4	− 0·106 74	− 0·108 81	− 0·106 69
ω_0	23° 27′ 51·95″	23° 27′ 53·81″	23° 27′ 54·22″	23° 27′ 54·80″	23° 27′ 55·61″	23° 27′ 54·80″
1 000 f	+ 0·009 84	+ 0·009 84	+ 0·007 35	+ 0·007 05	+ 0·007 19	+ 0·006 41
P	+ 50·111 36″	+ 50·223 50″	+ 50·241 1 ″	+ 50·222 95″	+ 50·224 43″	+ 50·225 20″
1 000 P′	+ 0·122 15	+ 0·122 15	+ 0·113 4	+ 0·112 07	+ 0·112 89	+ 0·110 22
Q	− 0·521 41″	− 0·483 95″	− 0·473 8 ″	− 0·467 70″	− 0·475 51″	− 0·466 47″
1 000 Q′	− 0·002 72	− 0·002 72	− 0·001 4	− 0·001 40	− 0·001 49	− 0·000 73

As Dr. PETERS's formulæ for nutation have been more used than any others, it will suffice to compare our own with them. Accordingly, formulæ (242) and (243) have been reduced to the epoch 1800, and are given below, side by side with Dr. PETERS's more elaborate expressions.* When extreme accuracy is desired, some of his small terms may advantageously be added to our formulæ.

Formulæ (242)	Dr. PETERS.	
$\Psi = -17{\cdot}229\,5'' \sin ☊$	$-17{\cdot}240\,5'' \sin ☊$	
$+\ 0{\cdot}207\,0 \sin 2☊$	$+\ 0{\cdot}207\,3 \sin 2☊$	
$-\ 0{\cdot}204\,2 \sin 2☾$	$-\ 0{\cdot}204\,1 \sin 2☾$	
$+\ 0{\cdot}068\,6 \sin A_{☾}$	$+\ 0{\cdot}067\,7 \sin (☾ - \Gamma')$	
	$-\ 0{\cdot}033\,9 \sin (2☾ - ☊)$	
	$+\ 0{\cdot}012\,5 \sin (2⊙ - ☊)$	(277)
	$-\ 0{\cdot}026\,1 \sin (3☾ - \Gamma')$	
	$+\ 0{\cdot}011\,5 \sin (☾ + \Gamma')$	
	$+\ 0{\cdot}015\,0 \sin (☾ - 2⊙ + \Gamma')$	
	$+\ 0{\cdot}005\,8 \sin (☾ + ☊ - \Gamma')$	
	$+\ 0{\cdot}005\,7 \sin (☾ - ☊ - \Gamma')$	
	$+\ 0{\cdot}002\,0 \sin (☾ - ☊ + \Gamma')$	

* From 109, pp. 170–172.

	$+\ 0.0044\ \sin(2\Gamma' - ☊)$	
	$+\ 0.0061\ \sin(2☾ - 2\odot)$	
	$-\ 0.0052\ \sin(3☾ - 2\odot + \Gamma')$	(277) cont'd.
	$+\ 0.0053\ \sin(2\odot - 2\Gamma')$	
	$+\ 0.0026\ \cos\Gamma'$	
	$+\ 0.0020\ \sin 2\Gamma'$	
	$+\ 0.0025\ \sin(☾ + 2\odot - \Gamma')$	
	$+\ 0.0028\ \sin(2☾ - 2\Gamma')$	
	$+\ 0.0024\ \sin(2☾ - 2☊)$	
	$-\ 0.0024\ \sin(2\odot - 2☊)$	
	$-\ 0.0028\ \sin(4☾ - 2\Gamma')$	
	$-\ 0.0033\ \sin(4☾ - 2\odot)$	
$-\ 1.2656\ \sin 2\odot$	$-\ 1.2694\ \sin 2\odot$	
$+\ 0.1274\ \sin A_{\odot}$	$+\ 0.1279\ \sin(\odot - \Gamma)$	
	$-\ 0.0213\ \sin(\odot + \Gamma)$	
	$-\ 0.0058\ \sin(3\odot - \Gamma)$	
	$-\ 0.0005\ \sin(2\odot - 2\Gamma)$	

Formulæ (243)	Dr. PETERS.	
$\Omega = +9.2158'' \cos ☊$	$+9.2231'' \cos ☊$	
$-0.0898\ \cos 2☊$	$-0.0897\ \cos 2☊$	
$+0.0886\ \cos 2☾$	$+0.0886\ \cos 2☾$	
	$+0.0181\ \cos(2☾ - ☊)$	
	$-0.0067\ \cos(2\odot - ☊)$	
	$+0.0113\ \cos(3☾ - \Gamma')$	
	$-0.0050\ \cos(☾ + \Gamma')$	
	$-0.0031\ \cos(☾ + ☊ - \Gamma')$	
	$+0.0030\ \cos(☾ - ☊ - \Gamma')$	
	$-0.0010\ \cos(☾ - ☊ + \Gamma')$	(278)
	$-0.0024\ \cos(2\Gamma' - ☊)$	
	$+0.0023\ \cos(3☾ - 2\odot + \Gamma')$	
	$+0.0023\ \sin\Gamma'$	
	$-0.0008\ \cos 2\Gamma'$	
	$-0.0011\ \cos(☾ + 2\odot - \Gamma')$	
	$+0.0012\ \cos(4☾ - 2\Gamma')$	
	$+0.0014\ \cos(4☾ - 2\odot)$	
$+0.5493'' \cos 2\odot$	$+0.5510\ \cos 2\odot$	
	$+0.0093\ \cos(\odot + \Gamma)$	
	$+0.0027\ \cos(3\odot - \Gamma)$	

The differential formulæ given above will suffice for computing the precession in all ordinary cases; but if the general formulæ should be required, their computation may be greatly facilitated by using the convenient expressions given by Professor STOCKWELL for the constants which determine the secular inequalities of the nodes and inclinations of the orbits of the eight principal planets of the solar system.*

* S4, pp. 161–164 and 171–176.

24.—THE DENSITY, FLATTENING, AND MOMENTS OF INERTIA OF THE EARTH.

In dealing with the equations (291), and in discussing the law of density of the interior of the Earth, we shall need the best attainable values of the Earth's mean density, its surface density, and its precessional moment of inertia.

The following is believed to be a tolerably complete list of the more important determinations of the Earth's mean density hitherto published:

1778.	MASKELYNE and HUTTON, from measurements of the attraction of the plumb line by the mountain Schehallien, in Scotland. (290, p. 783.)	4 ½
1798.	CAVENDISH, from the attraction of two leaden balls, each weighing 348·4 pounds, the attraction being measured by a torsion balance. (280, p. 522.)	5·48 ± 0·39
1811.	PLAYFAIR, from his own determination of the mass of Schehallien, combined with MASKELYNE and HUTTON's determination of its attractive force. (302, p. 376.)	4·713
1812.	HUTTON, from MASKELYNE's measurements of the attractive force of Schehallien, combined with PLAYFAIR's data for its density. (291, p. 64.)	5·
1823.	CARLINI, from pendulum observations made on Mt. Cenis, and at Bordeaux at the level of the sea. (279, p. 40.)	4·39
1837.	REICH, from the attraction of a leaden ball, measured by a torsion balance. (271, p. 98.)	5·438 ± 0·023
1840.	GIULIO, from CARLINI's pendulum observations on Mt. Cenis, after correcting both CARLINI's theory and his adopted length of the pendulum at Bordeaux. (287, p. 384.)	4·95
1842.	SAIGEY, from the pendulum observations made by BOUGUER and LA CONDAMINE in Peru, on Chimborazo and at the level of the sea, in 1737–1740. (272, parte 2nda, p. 125.)	4·62
1842.	BAILY, from the attraction of two spheres of lead, each weighing 380·5 pounds, the attraction being measured by a torsion balance. (271, p. ccxlvij.)	5·675 ± 0·004
1851.	REICH, from a rediscussion of the experiments which he made in 1837. (310, p. 389.)	5·484 ± 0·020
1851.	REICH, from the attraction of a leaden ball, measured by a torsion balance during the years 1847–1849 and 1850. (310, p. 418.)	5·583 ± 0·015
1856.	JAMES and CLARKE, from the attraction of the plumb line at Arthur's Seat, Scotland. (282, p. 606.)	5·316 ± 0·054
1856.	AIRY, from his pendulum experiments in the Harton Colliery, England. (270, pp. 342 and 355.)	6·566 ± 0·018
1856.	HAUGHTON, from AIRY's pendulum experiments in the Harton Colliery. (288, p. 51.)	5·480
1865.	PECHMANN, from deviations of the plumb line in the Alps, 4·7, 5·32, and 6·13; the mean of which is (14, T. 2, p. 380.)	5·38
1878.	CORNU and BAILLE, from measurements, with a torsion balance, of the attraction of a mass of mercury weighing about 26 pounds. (284, p. 699.)	5·56
1878.	CORNU and BAILLE, from BAILY's experiments, after correcting a systematic error. (284, p. 702.)	5·559
1878.	POYNTING, by weighing with a delicate balance the attraction of a sphere of lead having a mass of 340 pounds. (305, p. 18.)	5·69 ± 0·15
1881.	MENDENHALL, from pendulum observations at Tokio, and upon the summit of Fujiyama, in Japan. (299, p. 124, and 300, p. 103.)	5·77
1883.	STERNECK, from pendulum experiments made in the mines at Příbram, Bohemia. (314, p. 91. Mr. STERNECK has made similar experiments in other mines, with very anomalous results. For a review of them, see 308, pp. 234–237.)	5·77
1883.	VON JOLLY, from the change in the relative weight of two spherical bottles of mercury, each having a mass of 11·0 pounds, when they were compared, 1st, with both bottles close to the scale pans, and, 2d, with one bottle close to its pan, and the other suspended 68·9 feet lower down. (293, p. 22.)	5·692 ± 0·068
1889.	WILSING, from the attraction exerted upon a pendulum by two cylindrical masses of iron, each weighing 715 pounds. (322, p. 141.)	5·579 ± 0·012

In several instances two or more of the above results rest upon a single set of experiments, the reductions having been made by different methods, and in such cases we have only to consider the one which seems most trustworthy. After the application of that precept, the various results may be classified according to the methods of observation, as follows:

1.—Results from the attractions of mountains, measured by the deviations of the plumb line.

1811. MASKELYNE, HUTTON, and PLAYFAIR	4·713	
1856. JAMES and CLARKE	5·316	(279)
1865. PECHMANN	5·38	
Arithmetical mean	5·136	

2.—Results from the attractions of known masses of metal, measured either by the torsion balance or by the pendulum.

1798. CAVENDISH	5·48	
1851. REICH, from his experiments in 1837	5·484	
1851. REICH, from his experiments made in 1847–'50	5·583	(280)
1878. CORNU and BAILLE	5·56	
1878. BAILY, corrected by CORNU and BAILLE . . .	5·559	
1889. WILSING	5·579	
Arithmetical mean	5·541	

3.—Results from comparisons of pendulum observations made at different distances from the center of the Earth; namely, at the sea-level and upon mountains, or at the surface of the Earth and down in mines.

1840. CARLINI, corrected by GIULIO	4·95	
1842. BOUGUER, LA CONDAMINE, and SAIGEY . . .	4·62	
1856. AIRY	6·566	(281)
1881. MENDENHALL	5·77	
1883. STERNECK	5·77	
Arithmetical mean	5·535	

4.—Results from weighings made with delicate balances of the usual form.

1878. POYNTING	5·69	(282)
1883. VON JOLLY	5·692	
Arithmetical mean	5·691	

The arithmetical mean of the 16 values of the density of the Earth given in (279), (280), (281), and (282) is 5·482, but that attributes too much weight to the discordant values in (279) and (281). On account of our utter ignorance respecting the internal constitution of the Earth, it will be safer to base our conclusions solely upon experiments of the second and fourth kinds; and if we take the arithmetical mean of the eight values in (280) and (282), we shall have

$$\text{Mean density of the Earth} = 5·578 \pm 0·019$$

The last four values of (280) are unquestionably the most trustworthy of the whole series, and the arithmetical mean of them alone gives

$$\text{Mean density of the Earth} = 5{\cdot}570 \pm 0{\cdot}004$$

Probably it will be best to take the mean of the eight values in (280) and (282), giving half weight to the first two in (280) and to those in (282). In that way we find

$$\text{Mean density of the Earth} = 5{\cdot}576 \pm 0{\cdot}016 \qquad (283)$$

which will be adopted. It is scarcely necessary to add that the unit of density here employed is that of distilled water at a temperature of 39·2° Fahrenheit.

As a basis for estimating the surface density of the Earth, we have the following numbers:

1811. PLAYFAIR'S data give for the mean density of Schehallien, 9 933 × 4·713 — 17 804 (302, pp. 374 and 376)	2·63
1823. CARLINI, from the lithological constitution of Mt. Cenis, estimated its mean density to be (279, p. 39) . .	2·66
1852. PLANA found, from CARLINI'S pendulum experiments, for the mean density of the rocks upon which the plateau of Mt. Cenis rests (301, p. 187)	2·71
1856. JAMES and CLARKE found for the mean density of Arthur's Seat, Scotland (282, p. 603)	2·75
1856. AIRY'S data give for the crust in the neighborhood of Harton Colliery, England (270, p. 342)	2·53
1882. STERNECK found for the mean density of the crust over the Přibram mines, Bohemia (313, p. 118) . . .	2·75
1889. F. W. CLARKE found for the mean density of the outer ten miles of the Earth's crust	2·40

The thanks of the present writer are due to Professor CLARKE for his estimate, which was kindly communicated in the appended letter:

WASHINGTON, D. C., *November* 11, 1889.

DEAR PROF. HARKNESS:

In my estimates relative to the abundance of the chemical elements, I have assumed, for definiteness, a layer of the Earth's crust ten miles thick below sea-level. The volume of this, including the ocean and the continents above sea-level, is 1 935 000 000 cubic miles, of which 1 633 000 000 is solid, and 302 000 000 sea. Hence, by volume, in round numbers we have 85 per cent. rock and 15 per cent. water. The density of the ocean is a trifle under 1·03. The figure 1·03, then, may be taken as near enough for practical purposes.

The density of the solid crust is less easy to determine. The greater part of that crust is probably made up of plutonic rocks; and the average specific gravity of about 200 of these, representing a wide range of localities and varieties, is 2·716. In the crust are both heavier and lighter inclusions, and at its surface we have bodies of rather less heavy sedimentary rocks, which range down to a specific gravity of 2·5, or lower. Probably 2·60 or 2·65 would be near enough for the whole solid mass.

Now, taking 85 per cent. solid and 15 per cent. liquid, putting the latter at 1·03 density, we get the following data for the mean density of the whole mass:

With density of solid crust 2·5 . .	2·279 5
With density of solid crust 2·6 . .	2·364 5
With density of solid crust 2·7 . .	2·449 5

Total difference, 0·17, or *about* 7·5 per cent. of the lowest value. Probably 2·4, with an uncertainty of 4 per cent., may be assumed as approximately correct.

Yours, truly,

F. W. CLARKE.

The arithmetical mean of the first six of the above estimates is 2·67, and if we accept that as the mean density of the solid part of the Earth's crust, and assume, in accordance with Professor CLARKE's figures, that 0·844 of the crust is solid and 0·156 liquid, then the mean density of the crust will be

$$2·67 \times 0·844 + 1·03 \times 0·156 = 2·41$$

which agrees closely with Professor CLARKE's own estimate.

But we must not forget the many facts which seem to indicate that the Earth's crust is approximately in a state of hydrostatic equilibrium, and lead to the conclusion that it is more dense beneath the ocean than in the continents.* If such is really the case the comparative lightness of the waters of the sea must be at least partially compensated for by the increased density of the strata upon which they rest, and the average surface density of the Earth must exceed Professor CLARKE's estimate. Our present knowledge is too meager to warrant any very definite conclusion, but as the continental surface density probably lies between 2·40 and 2·72, we may take the mean of these numbers and attribute to it a probable error equal to half their difference. In that way we find

$$\text{Surface density of the Earth} = 2·56 \pm 0·16 \qquad (284)$$

The phenomena of precession and nutation, and certain perturbations of the Moon, enable us to determine two independent functions of the Earth's moments of inertia, from the first of which the Earth's flattening could be found if the distribution of density in its interior were known, while from the second it can be found without that knowledge. In employing the first function LE GENDRE's law of the distribution of density is usually assumed, and we have now to examine the process thus arising, and the result to which it leads.

If we put A, B, and C for the three principal moments of inertia of the Earth, A being the least and C the greatest, then the *precessional moment of inertia* will be $(2C - A - B)/2C$. Some writers have called this "the terrestrial constant of precession and nutation," or even "the precession constant," but that is certainly objectionable, because these words have long been employed to designate the annual motion of the vernal equinox, and their use for any other purpose can only lead to needless confusion. SERRET gives†

$$\frac{2C - A - B}{2C} = \frac{2n}{3m^2}\varkappa$$

where m and n are respectively the sidereal angular velocities of the Earth about the

*286½, p. 364. †83, p. 324.

Sun, and about its own axis. Accordingly, with the Julian year as unit of time, and the value of m from page 78

$$n/m = 366{\cdot}249\,983 \qquad m = 6{\cdot}283\,075\,94$$

$$\frac{2C - A - B}{2C} = 38{\cdot}861\,007\,\varkappa \tag{285}$$

or, if A and B are assumed to be equal, and $\varkappa$ is expressed in seconds of arc

$$\frac{C - A}{C} = 0{\cdot}000\,188\,403\,48\,\varkappa \tag{286}$$

From (230), after substituting the numerical values of its A, B, C, and cos ω_0 for the epoch 1850·0

$$\varkappa = +\,1{\cdot}089\,640\,1\mathfrak{L} - 4{\cdot}071\,361\,5\mathfrak{A} \tag{287}$$

and by substituting (287) in (286)

$$\frac{C - A}{C} = 0{\cdot}000\,205\,292\,0\mathfrak{L} - 0{\cdot}000\,767\,058\,7\mathfrak{A} \tag{288}$$

Formula (286) is perfectly general, but (287) and (288) apply rigorously only to the epoch 1850, and consequently $\mathfrak{L}$ and $\mathfrak{A}$ must be reduced to that epoch before being employed in them.

From (231), $\varkappa = 17{\cdot}348\,66''$, whence by (286)

$$\frac{C - A}{C} = 0{\cdot}003\,268\,55 = \frac{1}{305{\cdot}95} \tag{289}$$

For convenience of reference, some of the values which other investigators have found for $\varkappa$ and (C − A)/C are given in Table VIII. That attributed to La Place is what he himself computed, but his values of the precession and nutation constants are respectively 50·261″ and 10·06″ when reduced to 1850·0, and their substitution in (288) gives a somewhat greater result, namely (C − A)/C = 0·002 601 61″. Bessel

Table VIII.—*Values of* $\varkappa$ *and* (C − A)/C *according to various Authors.*

Date.	Author.	κ	(C − A) : C
		″	
1799	La Place (7, T. 2, liv. 5, chap. 1, §§ 13–14)	. . .	0·002 596 62 = 1 : 385·12
1818	Bessel (1, p. 130)	. . .	·002 924 50 = 1 : 341·93
1830	Bessel (1½, pp. v and xv)	18·312	·003 450 16 = 1 : 289·84
1841	C. A. F. Peters (109, p. 101)	17·362	·003 271 12 = 1 : 305·71
1856	Le Verrier (8, T. 2, p. 174)	17·323	·003 263 77 = 1 : 306·39
1859	Serret (83, p. 323)	17·378	·003 274 13 = 1 : 305·42
1862	Hansen (55, p. 472)	. . .	·003 272 = 1 : 305·62
1889	Formulæ (231) and (289)	17·349	0·003 268 55 = 1 : 305·95

did not explain how he obtained the result quoted in Table VIII from the Fundamenta. The precession and nutation constants employed in that work are respectively 50·316 14″ and 9·648 9″, for the epoch of 1850·0, and their substitution in (288) would give $(C-A)/C = 0\cdot002\,928\,28$. The Tabulæ Regiomontanæ contains neither $\varkappa$ nor $(C-A)/C$, but its values of the precession and nutation constants are respectively 50·351 36″ and 8·977 97″ for 1850·0, and their substitution in (287) and (288) leads to the values given in Table VIII. LE VERRIER gave only $\varkappa$, from which $(C-A)/C$ has been computed by means of formula (286).

Taking the polar semi-diameter of the Earth for unity, if we put b for the polar semi-axis of any one of the hypothetical equipotential strata composing the Earth, and ρ and ε for the density and ellipticity of the same stratum, then CLAIRAUT's derived equation may be written*

$$0 = \left(b^2\frac{d^2\varepsilon}{db^2} - 6\varepsilon\right)\int_0^b \rho b^2 db + 2\left(b\frac{d\varepsilon}{db} + \varepsilon\right)\rho b^3 \tag{290}$$

and from its integration the theoretical relations of the quantities discussed above will result. But in order to effect the integration it is first necessary to express ρ in terms of b; or, in other words, the law governing the distribution of density in the interior of the Earth is required. LE GENDRE's (commonly called LA PLACE's) law has usually been assumed, and the investigation has been put in various forms by different authors.† We shall employ the expressions given by THOMSON and TAIT,‡ viz.

$$\begin{aligned} f &= \frac{3}{\theta^2}(1-\theta\cot\theta) \\ \frac{5\sigma_0}{2\varepsilon} &= \frac{f\theta^2}{3(f-1)} - \frac{3}{f} \\ \frac{C-A}{C} &= \frac{\varepsilon - \frac{1}{2}\sigma_0}{1-6(f-1)/f\theta^2} \end{aligned} \tag{291}$$

where f is the ratio of the Earth's mean density to its surface density; σ_0 is the ratio of the centrifugal force to gravity, both taken at the equator; and ε is the Earth's flattening.

These formulæ afford the means of deriving ε and f from the observed value of $(C-A)/C$; but here we encounter the difficulty that θ is the real independent variable, and any attempt to change it leads to very complicated algebraic expressions. To avoid them Table IX has been formed, in which the numerical values of all the quantities involved in the equations (291) are exhibited throughout a sufficient range of the argument θ. In deriving ε from $5\sigma_0/2\varepsilon$ we have taken

$$\sigma_0 = \frac{4a}{lt_1^2} = 0\cdot003\,467\,833 = \frac{1}{288\cdot364\,5} \tag{292}$$

where $t_1^2 = 7\,424\,252\,068^s$, $a = 20\,926\,202$ feet, $l = 3\cdot251\,169$ feet, from (3), (13), and (17).

* 281, p. 276 and 321, vol. 1, pp. 225, 226.

† 294, p. 408; 321, vol. 2, p. 117; 7, T. 2, liv. 5, chap. 1, § 14 and T. 5, liv. 11, chap. 2, § 6; 10, T. 2, p. 472; 17, p. 235; 24½; 15½, pp. 111 and 149; 13, pp. 83–87; 14, Teil 2, p. 487.

‡ 11, vol. 1, part 2, pp. 407 and 414.

Entering Table IX with the observed value of $(C-A)/C$ from (289), viz, 0·003 268 60, we find

$$\begin{aligned}\theta &= 144{\cdot}652\,9^\circ \qquad f = 2{\cdot}145\,96 \\ \varepsilon &= 0{\cdot}003\,359\,4 = 1 : 297{\cdot}67\end{aligned} \tag{293}$$

Whence, with the value of the Earth's mean density from (283)

$$\text{Surface density of the Earth} = 5{\cdot}576/2{\cdot}146 = 2{\cdot}598 \tag{294}$$

TABLE IX.—*Numerical Values of the Quantities which enter the Equations* (291).

$\frac{\theta}{\pi}180^\circ$	θ	f	$\frac{5\sigma_0}{2\varepsilon}$	ε	$\frac{C-A}{C}$
134°	2·338 74	1·787 20	2·460 73	1 : 283·835 = 0·003 523 17	1 : 288·851 = 0·003 461 99
135	2·356 19	1·813 56	2·470 87	1 : 285·004 = 0·003 508 72	1 : 290·259 = 0·003 445 20
136	·373 65	·841 19	·481 22	286·198 494 08	291·703 428 14
137	·391 10	·870 14	·491 79	287·418 479 25	293·186 410 80
138	·408 55	·900 48	·502 59	288·664 464 24	294·708 393 19
139	·426 01	·932 29	·513 63	289·936 449 04	296·271 375 29
140	2·443 46	1·965 69	2·524 90	1 : 291·237 = 0·003 433 63	1 : 297·875 = 0·003 357 11
141	·460 91	2·000 80	·536 41	292·565 418 04	299·523 338 64
142	·478 37	·037 75	·548 18	293·922 402 26	301·217 319 87
143	·495 82	·076 71	·560 21	295·310 386 27	302·957 300 80
144	·513 27	·117 85	·572 50	296·727 370 09	304·746 281 42
145	2·530 73	2·161 36	2·585 06	1 : 298·176 = 0·003 353 72	1 : 306·586 = 0·003 261 73
146	·548 18	·207 46	·597 91	299·658 337 14	308·479 241 71
147	·565 63	·256 36	·611 05	301·174 320 34	310·428 221 36
148	·583 09	·308 32	·624 49	302·724 303 34	312·435 200 67
149	·600 54	·363 60	·638 24	304·310 286 12	314·501 179 64
150	2·617 99	2·422 49	2·652 31	1 : 305·933 = 0·003 268 69	1 : 316·630 = 0·003 158 26

LE GENDRE's law of the distribution of density within the Earth is given by the equation*

$$\rho = \frac{F}{r}\sin\frac{r}{\varkappa} = \frac{F}{r}\sin\theta \tag{295}$$

where ρ is the density at the distance r from the Earth's center, F and $\varkappa$ are constants, and $\theta = r/\varkappa$. Putting r', ρ', and θ' for the surface values of r, ρ, and θ, and taking r' for unity, we have from (293), (294), and (295)

$$F = \rho'/\sin\theta' = 4{\cdot}490\,7$$

Further, if n is a fraction such that $nr' = r$, then $nr'/\varkappa = n\theta'$; and therefore, upon substituting in it the numerical values of F and θ', equation (295) takes the form

$$\rho = \frac{4{\cdot}490\,7}{n}\sin(n\,144{\cdot}652\,9^\circ) \tag{296}$$

*11, vol. 1, part 2, p. 404.

To evaluate ρ when n becomes zero, (295) must be written

$$\rho = \frac{\sin \frac{r}{n}}{\frac{r}{F}}$$

whence, by differentiating both numerator and denominator, and making r zero, we find for the center of the Earth

$$\rho = \frac{\frac{1}{n} \cos \frac{r}{n}}{\frac{1}{F}} = \frac{F}{n} = \theta' F \qquad (297)$$

Table X exhibits the values given by formulæ (296) and (297) for the density of the Earth at various distances from its center; distilled water at 39·2° Fahrenheit being taken as unity.

TABLE X.—*Density of the Interior of the Earth according to* LE GENDRE'S *Law.*

Distance from center.	Density.	Distance from center.	Density.
1·0	2·598	0·4	9·506
0·9	3·812	0·3	10·284
0·8	5·057	0·2	10·862
0·7	6·292	0·1	11·217
0·6	7·473	0·0	11·328
0·5	8·558		

Upon comparing the values of the flattening and surface density in (293) and (294) with the corresponding observed values in (4) and (284), viz:

$$\begin{aligned} \varepsilon &= 0{\cdot}003\,407\,5 = 1 : 293{\cdot}47 \\ \text{Surface density of the Earth} &= 2{\cdot}56 \pm 0{\cdot}16 \end{aligned} \qquad (298)$$

a satisfactory agreement is found only in the case of the surface density. The two values of the flattening differ largely, but our knowledge respecting the figure of the Earth is scarcely sufficient to render it certain that the discordance exceeds the possible effect of errors of observation. Although the value of ε in (293) does not agree with that derived by General CLARKE from his discussion of the great geodetic arcs, it is nevertheless within the limits of the values found from pendulum experiments, as will be evident from an examination of Table XI.

TABLE XI.—*Flattening of the Earth as found from Pendulum Experiments.*

Date.	Author.	ε
1830	AIRY (17, p. 231)	1 : 282·89
1834	BAILY (29, p. 94)	1 : 285·26
1842	BORENEUS (29¾, p. 18)	1 : 289·
1853	PAUCKER (25½, T. 13, p. 230)	1 : 288·38
1869	UNFERDINGER (39, pp. 313, 324, and 329).	1 : 299·15
1876	A. FISCHER (24¼, p. 87)	1 : 284·4
1880	CLARKE (13, p. 350)	1 : 292·2 ± 1·5
1884	HELMERT (14, Teil 2, p. 241)	1 : 299·26 ± 1·26
1884	HILL (57⅓, p. 339, foot-note) :	
	For the northern hemisphere	1 : 285·44
	For the southern hemisphere	1 : 290·02

$(C-A)/C$ is certainly much better known than ε, and more data for the determination of the latter are greatly needed. Respecting its derivation from pendulum experiments, we may remark that when the length of the pendulum is expressed by a complicated formula, such as UNFERDINGER's or HILL's, the simplest procedure will be to compute the numerical length of the pendulum at the equator and at the pole, and then, calling these lengths respectively l_0 and l_{90}, formula (11) gives

$$\varepsilon = 1 - \frac{1}{l_0}\left(l_{90} - \frac{10a}{t_1^2}\right)$$

When the pendulum formula gives different lengths at the two poles, different flattenings will result for the two hemispheres. Perhaps something might be gained by using AIRY's extension of CLAIRAUT's theorem to terms of the second order.*

If Δ is the mean density of the Earth; 1, ρ', and ε' the values assumed by b, ρ, and ε at the sea-level; and $\rho = f(b)$; then it can be shown that after integration (290) must satisfy the following conditions:†

$$\text{I} \qquad f'(b) < 0$$

$$\text{II} \qquad \int_0^1 b^2 f(b)\,db = \tfrac{1}{3}\Delta$$

$$\text{III} \qquad f(1) = \rho'$$

$$\text{IV} \qquad \frac{\int_0^1 b^2 f(b)\,db}{\int_0^1 b^4 f(b)\,db} = \frac{C-A}{C(\varepsilon' - \frac{1}{2}\sigma_0)} \qquad (299)$$

$$\text{V} \qquad \frac{z_1\Delta}{2f(0) - 3\int_0^1 f(b)\frac{dz}{db}\,db} = \frac{2\varepsilon'}{5\sigma_0}$$

*16½, p. 562. †320, p. 523.

With our own numerical values from (4), (283), (284), and (292)

$$
\begin{aligned}
\tfrac{1}{3}\Delta &= 1.859 \pm 0.005\,3 \\
\rho' &= 2.56 \pm 0.16 \\
\frac{C-A}{C(\varepsilon' - \frac{1}{2}\sigma_0)} &= 1.953\,0 \\
\frac{2\varepsilon'}{5\sigma_0} &= 0.393\,05
\end{aligned}
\tag{300}
$$

ROCHE,* LIPSCHITZ,† DARWIN,‡ TISSERAND,§ HILL,|| and LÉVY¶ have used forms of $f(b)$ other than LE GENDRE's law for integrating equation (290), but, singularly enough, when substituted in the left-hand member of IV of (299) they all give very nearly 1.987, while with the geodetic value of ε the right-hand member gives 1.953. POINCARÉ has recently investigated the subject in a more general manner** and has concluded that no form of $f(b)$ which is continuous between the limits 0 and 1 can satisfy the observed values of precession and nutation together with General CLARKE's value of the flattening. He adds incidentally that the limiting values of the left-hand member of IV of (299) are 1.987 and 2.04; whence it follows that the limiting values of ε are 1 : 296 and 1 : 300. We are therefore in this dilemma: Either the flattening must lie between 1 : 296 and 1 : 300, or the distribution of density within the Earth can not be represented by any function which increases continuously from the surface to the center.

Those perturbations of the Moon which arise from the figure of the Earth have been discussed by many geometers, among whom LA PLACE, PLANA, PONTÉCOULANT, HANSEN, and HILL are conspicuous. If we designate the maximum values of the perturbations in latitude and longitude respectively by δs and $\delta\nu$, then according to HILL††

$$
\delta s = \frac{\beta_2}{a_2^2} G \qquad\qquad \delta\nu = \frac{\beta_2}{a_2^2} H \tag{301}
$$

where

$$
\begin{aligned}
G = &+\left(-\frac{2}{3} + \frac{40}{3}\gamma^2 + \frac{2}{3}e_2^2 + e_1^2 - \frac{13}{2}\gamma^4 + \frac{10}{3}\gamma^2 e_2^2 - \frac{267}{96}e_2^4\right. \\
&\left. - 20\gamma^2 e_1^2 - e_1^2 e_2^2 - \frac{1}{4}e_1^4 + \frac{5}{4}\frac{a_2^2}{a_1^2} - \frac{8}{9}\frac{\psi}{m^2 n''}\right)\frac{1}{m^2} \\
&+\left(-\frac{1}{4} + \frac{9}{2}\gamma^2 + 6e_2^2 + \frac{1}{9}e_1^2 - \frac{2}{3}\frac{\psi}{m^2 n''}\right)\frac{1}{m} - \frac{43}{18} + \frac{13223}{288}\gamma^2 \\
&+ \frac{30925}{1152}e_2^2 - \frac{19}{4}e_1^2 - \frac{3449}{576}m - \frac{59245}{3456}m^2
\end{aligned}
\tag{302}
$$

$$
\begin{aligned}
H = &+\left(+\frac{38}{3}\gamma - 7\gamma^3 - \frac{20}{3}\gamma e_2^2 - 19\gamma e_1^2 + \frac{152}{9}\frac{\gamma\psi}{m^2 n''}\right)\frac{1}{m^2} \\
&+\left(+\frac{13}{4}\gamma - \frac{135}{8}\gamma^3 - 88\gamma e_2^2 - \frac{13}{9}\gamma e_1^2\right)\frac{1}{m} + \frac{13585}{288}\gamma + \frac{5825}{576}\gamma m
\end{aligned}
\tag{303}
$$

*311. †296 and 297. ‡286. §317 and 319. ||289. ¶295. **303, p. 67. ††57⅓, pp. 213, 308, and 316.

With the numerical values of $m, n'', \gamma, e_1, e_2, \psi$, and a_2/a_1 from (54), (33), (64), (24), (61), (241), (168), and Table VI, (302) and (303) give

$$G = -119.770\,97 \qquad H = +105.290\,95 \tag{304}$$

Further

$$\alpha_2 = \frac{a(1+\varkappa')}{\sin P} \qquad \beta_2 = \frac{3 \sin 2\varpi}{2E'}\left(C - \frac{A+B}{2}\right)$$

and by substituting these values in (301) we have, when A and B are assumed equal, and δs and $\delta \nu$ are expressed in seconds of arc

$$\frac{3 \sin^2 P \sin 2\varpi}{2E'a^2(1+\varkappa')^2 \operatorname{arc} 1''}(C - A) = \frac{\delta s}{G} = \frac{\delta \nu}{H} \tag{305}$$

Whether the Earth was originally fluid or not, for our present purpose it may be regarded as covered by a fluid, because all observations relating to its figure are reduced to the sea-level; and by combining that fact with the single assumption that its interior is composed of nearly spherical strata whose densities are any function whatever of their distances from the center, La Place showed that*

$$2C - A - B = \frac{16}{3}\pi(\varepsilon - \tfrac{1}{2}\sigma_0)a^2\int \rho r^2 dr$$

whence, as $\int \rho r^2 dr = \frac{E'}{4\pi}$, and A and B are assumed to be equal

$$C - A = \frac{2}{3}(\varepsilon - \tfrac{1}{2}\sigma_0)E'a^2 \tag{306}$$

By eliminating C — A between (305) and (306), the following expressions result for finding the flattening of the Earth from the observed values of δs and $\delta \nu$:

$$\frac{\sin^2 P \sin 2\varpi}{(1+\varkappa')^2 \operatorname{arc} 1''}(\varepsilon - \tfrac{1}{2}\sigma_0) = \frac{\delta s}{G} = \frac{\delta \nu}{H} \tag{307}$$

With the numerical values from Table VI, (57), (78), (292), (302) and (303), (305) and (307) give

$$\begin{aligned} C - A &= -0.000\,134\,44\, E'\,a^2\delta s \\ &= +0.000\,152\,93\, E'\,a^2\delta \nu \end{aligned} \tag{308}$$

$$\begin{aligned} \varepsilon &= +0.001\,733\,92 - 0.000\,201\,67\,\delta s \\ &= +0.001\,733\,92 + 0.000\,229\,40\,\delta \nu \end{aligned} \tag{309}$$

On account of accidental errors in the observed values of δs and $\delta \nu$, the two formulæ in each of these pairs will seldom yield identical results. To obtain the most

*7, T. 3, liv. 7, chap. 2, § 20.

probable values of C — A and ε we must therefore resort to the method of least squares, and when the probable errors of δs and $\delta\nu$ are equal, that gives

$$\begin{aligned} C - A &= (-0{\cdot}000\,075\,836\,\delta s + 0{\cdot}000\,066\,668\,\delta\nu)E'a^2 \\ \varepsilon &= +0{\cdot}001\,733\,92 - 0{\cdot}000\,113\,75\,\delta s + 0{\cdot}000\,100\,00\,\delta\nu \end{aligned} \qquad (310)$$

The formulæ (309) are independent of all theories respecting the distribution of matter within the Earth, Sir Geo. G. Stokes having shown* that equation (306) remains valid, whatever that distribution may be; and, as Dr. Hill has carried the expressions (302) and (303) to terms of the eighth order in the lunar theory, the uncertainty in the coefficients of δs and $\delta\nu$ is not likely to exceed four or five units in the last place of decimals.

For comparison with (309), it will suffice to quote the formulæ found respectively by Pontécoulant and Hansen. Those of Pontécoulant are algebraically identical with (307), but his series for G and H are not pushed so far as in (302) and (303), the expressions employed being†

$$\begin{aligned} G &= \left(-\frac{2}{3} - \frac{m}{4} - \frac{43}{18}m^2 + e_1^2 + \frac{4}{3}\tan^2 I\right)\frac{1}{m^2} \\ H &= \left(+\frac{19}{3} + \frac{13}{8}m\right)\frac{\tan I}{m^2} \end{aligned}$$

With our values of the constants, from (24), (54), and (65), these expressions give

$$G = -121{\cdot}448\,40 \qquad H = +103{\cdot}880\,52$$

whence, by substitution in (307)

$$\begin{aligned} \varepsilon &= +0{\cdot}001\,733\,92 - 0{\cdot}000\,198\,88\,\delta s \\ &= +0{\cdot}001\,733\,92 + 0{\cdot}000\,232\,51\,\delta\nu \end{aligned} \qquad (311)$$

Hansen's formulæ are, in his own notation‡

$$\begin{aligned} \rho &= \tfrac{1}{2}\varphi - 2\left(\frac{a}{D}\right)^2 \frac{1 + 4\sin^2\frac{1}{2}J}{\sin\varepsilon\cos\varepsilon} \cdot \frac{\alpha + \eta - p}{2 + (\alpha + \eta - p)}\lambda_1 \\ \pi_1 &= 20\sin\tfrac{1}{2}J\left(1 + \frac{3}{2}\sin^2\tfrac{1}{2}J\right)\left[1 - \tfrac{1}{2}(\alpha + \eta - p)\right]\lambda_1 \end{aligned} \qquad (312)$$

His α, η, and p are related to our n_4'', μ, and ψ_1, through the expressions

$$n_4''(\alpha + \eta) = \mu \qquad n_4''p = \psi_1$$

and as his ρ, φ, D, J, ε, λ_1, and π_1, are our ε, σ_0, r_1, I, ω, δs, and $\delta\nu$, the formulæ (312) become in our notation

$$\begin{aligned} \varepsilon &= \tfrac{1}{2}\sigma_0 - \frac{(4 + 16\sin^2\frac{1}{2}I)\,\text{arc}\,1''}{\sin^2 P\sin 2\omega} \cdot \frac{\mu - \psi_1}{2n_4'' + \mu - \psi_1}\delta s \\ \delta\nu &= 20\sin\tfrac{1}{2}I(1 + \frac{3}{2}\sin^2\tfrac{1}{2}I)[1 - \tfrac{1}{2}(\mu - \psi_1)/n_4'']\delta s \end{aligned} \qquad (313)$$

*38½, p. 680. †10, T. 4, pp. 485–486. ‡55, pp. 348, 469, and 471.

With the numerical values from Table VI, (49), (53), (65), (78), (241) and (292), (313) gives

$$
\begin{aligned}
\delta\nu &= -0.898\,624\,\delta s \\
\varepsilon &= +0.001\,733\,92 - 0.000\,196\,62\,\delta s \\
&= +0.001\,733\,92 + 0.000\,218\,80\,\delta\nu
\end{aligned}
\qquad (314)
$$

The formulæ (309) and (311) are directly comparable with each other, but not with (314). Owing to the peculiar form of HANSEN's lunar theory, his perturbations differ from those found in the usual ecliptic theory, and to avoid the troublesome transformations which occur in passing from the one theory to the other, we shall derive the necessary transition factors by comparing (309) and (314). Distinguishing HANSEN's δs and $\delta\nu$ by accents, and equating the corresponding expressions in the two sets of formulæ, we have

$$
\begin{aligned}
201\,667\,\delta s &= 196\,617\,\delta s' \\
229\,400\,\delta\nu &= 218\,798\,\delta\nu'
\end{aligned}
$$

Whence

$$\delta s = 0.974\,96\,\delta s' \qquad \delta\nu = 0.953\,78\,\delta\nu' \qquad (315)$$

Table XII contains the values of δs and $\delta\nu$ determined from observation by various astronomers, together with the resulting values of C — A and ε, computed by means of the formulæ (308), (309), and (310). The values of δs and $\delta\nu$, which BÜRG originally derived in 1806, and revised in 1823–1825, are based upon 3233 observations of the Moon, made by MASKELYNE, at Greenwich, during the years 1765 to 1793 * BURCKHARDT's values are based upon "more than four thousand observations," which LA PLACE says were those of BRADLEY and MASKELYNE.† AIRY's value is based upon the entire series of lunar observations made at Greenwich during the years 1750 to 1851, in which the data used by BÜRG and BURCKHARDT are included, and constitute but a small part. HANSEN never explained the derivation of his values, but simply wrote: "Die Mondbeobachtungen haben mir gegeben $\lambda_1 = -8.382''$, $\pi_1 = -7.624''$," whence the numbers in Table XII result through the formulæ (315):

TABLE XII.—*Observed Values of certain Perturbations of the Moon which depend upon the Figure of the Earth, together with the resulting Values of* C — A *and* ε.

Date.	Authority.	δs	$\delta\nu$	C — A	ε
		$''$	$''$		
1806	BÜRG (39½, Introduction Tab. de la Lune) .	— 8·0	+ 6·8	0·001 060 03 E'/a^2	0·003 323 92 = 1 : 300·85
1812	BURCKHARDT (45¼, Introduction)	8·0	7·0	0·001 073 37 E'/a^2	0·003 343 92 = 1 : 299·05
1823	BÜRG (45½, s. 324)	8·6	. . .	0·001 138 20 E'/a^2	0·003 441 17 = 1 : 290·60
1825	BÜRG (45¾, s. 14)	. . .	7·29		
1861	AIRY (44, p. 12)	. . .	6·44	0·000 984 87 E'/a^2	0·003 211 26 = 1 : 311·40
1862	HANSEN (55, p. 470)	— 8·172	+ 7·272	0·001 104 54 E'/a^2	0·003 390 68 = 1 : 294·93

*45½, col. 314.

†57⅔, p. 225. LA PLACE was a member of the Commission appointed by the French Bureau of Longitudes to examine BURCKHARDT's tables, previous to their official adoption.

Of the results contained in Table XII, those found from the data of HANSEN are probably the most trustworthy. Accordingly we shall take

$$C - A = 0.001\,104\,54\,E'a^2 \tag{316}$$

$$\varepsilon = 0.003\,390\,68 = 1 : 294.93 \tag{317}$$

whence, by comparing (289) and (316)

$$A = 0.336\,83\,E'a^2 \qquad C = 0.337\,93\,E'a^2 \tag{318}$$

where E′ is the mass of the Earth, and a its equatorial semi-diameter.

This method of determining the numerical values of the Earth's moments of inertia is due to HANSEN,* who has pointed out that it affords a fine confirmation of the Earth's increase of density from its surface to its center, the value of C for a homogeneous spheroid being much greater, viz, 0.400 00 $E'a^2$.

25.—UNCERTAINTY IN THE VALUE OF ε, AND ITS EFFECT UPON THE OTHER CONSTANTS.

The actual state of our knowledge respecting the flattening of the Earth is best shown by bringing together the more important determinations of that quantity from (4), Table XI, (293) and (317). They are

From geodetic arcs (CLARKE)	1 : 293.47
From pendulum experiments (HILL)	1 : 287.71
From pendulum experiments (CLARKE)	1 : 292.2
From pendulum experiments (HELMERT)	1 : 299.26
From precession and nutation, combined with LE GENDRE's law of density	1 : 297.67
From perturbations of the Moon	1 : 294.93

Most unfortunately, every one of these results is affected by much uncertainty. General CLARKE used six separate arcs in determining his spheroid of 1880, all of which were well represented by equations (3) and (4). Recently two of the longest of these arcs have been connected, namely, the Anglo-French and the Russian, and according to statements made at the Paris International Geodetic Congress of October, 1889, it now appears that the similar ellipses passing through them have neither a common center nor common axes! Thus General CLARKE's ε is invalidated, and the possibility of deriving a trustworthy value from the present arcs becomes questionable. The pendulum experiments are equally unsatisfactory. The result which HELMERT deduced from them by the condensation method, differs largely from that found by HILL through a formula involving twenty unknown quantities, while General CLARKE's value, computed yet otherwise, agrees with neither HELMERT's nor HILL's, but lies midway between them. In 1827 BIOT discussed a number of pendulum experiments, which he divided into three groups, having their respective centers as nearly as possible at the

* 287½, s. 198.

equator, at latitude 45°, and at the pole. From the first and second groups he found $\varepsilon = 1 : 276{\cdot}38$; from the second and third, $\varepsilon = 1 : 306{\cdot}33$; and from the first and third, $\varepsilon = 1 : 290{\cdot}59$.* These results look as if the flattening varied with the region containing the dominating number of pendulum experiments, but it is more probable that their discordance arises from accidental peculiarities of the stations occupied, and there is reason to fear that even yet we have not accumulated sufficient data to eliminate all such peculiarities. As to the result derived from precession and nutation, it is evidently vitiated by our ignorance of the internal constitution of the Earth; and even the theoretically exact value from lunar perturbations is rendered questionable by the uncertainty attaching to the observed values of the perturbations themselves. Indeed the facts thus far adduced scarcely warrant any conclusion more definite than that the flattening probably lies between 1 : 290 and 1 : 300, but we shall see presently that there is some further evidence which tends in the direction of the smaller limit.

We have next to examine how much the system of constants given on page 73 is affected by the uncertainty in the Earth's flattening, and on account of the number of variables involved, the simplest procedure will be to recompute them all with an assumed flattening of 1 : 300. For that purpose the numerical coefficients in formulæ (156), (157), and (166) require modification.

A comparison of (152) and (156) shows that the logarithmic coefficient of the latter is

$$\tfrac{1}{3}\log\frac{4a(1+\varkappa)^3}{l\rho^2\,T^2(1+\sigma\sqrt{\tfrac{2}{3}})} - 4{\cdot}685\,574\,87 = 2{\cdot}784\,993\,22 \qquad (319)$$

Here l, ρ, and σ are functions of ε, and when ε becomes $\varepsilon + d\varepsilon$, l, ρ, and σ become respectively $l + dl$, $\rho + d\rho$, and $\sigma + d\sigma$. Accordingly, we may write

$$\frac{1}{(l+dl)(\rho+d\rho)^2\,[1+(\sigma+d\sigma)\sqrt{\tfrac{2}{3}}]} = \frac{1}{l\rho^2(1+\sigma\sqrt{\tfrac{2}{3}})} \times \frac{l}{l+dl} \times \frac{\rho^2}{(\rho+d\rho)^2} \times \frac{1+\sigma\sqrt{\tfrac{2}{3}}}{1+(\sigma+d\sigma)\sqrt{\tfrac{2}{3}}}$$

and, therefore,

$$\begin{aligned}\tfrac{1}{3}\log\frac{4a(1+\varkappa)^3}{(l+dl)(\rho+d\rho)^2\,T^2[1+(\sigma+d\sigma)\sqrt{\tfrac{2}{3}}]} - 4{\cdot}685\,574\,87 \\ = 2{\cdot}784\,993\,22 + \tfrac{1}{3}\log\frac{l}{l+dl} + \tfrac{1}{3}\log\frac{\rho^2}{(\rho+d\rho)^2} + \tfrac{1}{3}\log\frac{1+\sigma\sqrt{\tfrac{2}{3}}}{1+(\sigma+d\sigma)\sqrt{\tfrac{2}{3}}}\end{aligned} \qquad (320)$$

Formula (11) gives for the length of the seconds pendulum at latitude 45°

$$l_{45} = l_0\,(1 - \tfrac{1}{2}\varepsilon) + 10a/2t_1^2$$

whence

$$l_0 = \frac{l_{45} - 10a/2t_1^2}{1 - \tfrac{1}{2}\varepsilon} \qquad (321)$$

* 20½, p. 38.

Again, as arc sin $\sqrt{\tfrac{1}{3}} = 35^\circ\ 15'\ 52''$, the length of the pendulum at that latitude is

$$l_{35\cdot3} = l_0(1 - \tfrac{1}{3}\varepsilon) + \frac{10a}{3t_1^2} = \left(l_{45} - \frac{10a}{2t_1^2}\right)\frac{1 - \frac{1}{3}\varepsilon}{1 - \frac{1}{2}\varepsilon} + \frac{10a}{3t_1^2} \tag{322}$$

or, with the numerical values of l_{45}, a and t_1 from page 9

$$l_{35\cdot3} = 3\cdot245\,629\,7\,\frac{1 - \frac{1}{3}\varepsilon}{1 - \frac{1}{2}\varepsilon} + 0\cdot009\,395\,4 \text{ feet} \tag{323}$$

Differentiating (323), and dropping the subscript 35·3

$$dl = \frac{3\cdot245\,63}{6(1 - \frac{1}{2}\varepsilon)^2}d\varepsilon = \frac{0\cdot540\,94}{(1 - \frac{1}{2}\varepsilon)^2}d\varepsilon \tag{324}$$

and then, with sufficient accuracy

$$\frac{l}{l + dl} = 1 - \frac{dl}{l} = 1 - \frac{0\cdot540\,94}{l(1 - \frac{1}{2}\varepsilon)^2}d\varepsilon$$
$$\log\frac{l}{l + dl} = -\frac{0\cdot540\,94\,\mathrm{M}}{l(1 - \frac{1}{2}\varepsilon)^2}d\varepsilon \tag{325}$$

where M is the modulus of the common system of logarithms.

From (4) and (5)

$$e^2 = 2\varepsilon - \varepsilon^2 \qquad de = \frac{(1 - \varepsilon)\,d\varepsilon}{e} \tag{326}$$

and with that value of e^2, and $\sin^2\varphi = \frac{1}{3}$, (7) gives

$$\rho^2 = \frac{3 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4}{3 - 2\varepsilon + \varepsilon^2} \tag{327}$$

Differentiating

$$d\rho = \frac{-6 + 30\varepsilon - 44\varepsilon^2 + 28\varepsilon^3 - 10\varepsilon^4 + 2\varepsilon^5}{2\rho(3 - 2\varepsilon + \varepsilon^2)^2}d\varepsilon \tag{328}$$

and then, with sufficient accuracy

$$\frac{\rho^2}{(\rho + d\rho)^2} = \frac{\rho^2}{\rho^2\left(1 + \frac{2d\rho}{\rho}\right)} = 1 - \frac{2d\rho}{\rho}$$
$$= 1 + \frac{6 - 30\varepsilon + 44\varepsilon^2 - 28\varepsilon^3}{9 - 18\varepsilon + 29\varepsilon^2 - 36\varepsilon^3}d\varepsilon \tag{329}$$
$$\log\frac{\rho^2}{(\rho + d\rho)^2} = +\frac{6 - 30\varepsilon + 44\varepsilon^2 - 28\varepsilon^3}{9 - 18\varepsilon + 29\varepsilon^2 - 36\varepsilon^3}\mathrm{M}d\varepsilon$$

The differentiation of (148) with respect to σ, l, and e gives

$$d\sigma = \sigma\left\{\frac{e\sin^2\varphi}{1 - e^2\sin^2\varphi}de - \frac{dl}{l}\right\} \tag{330}$$

and by substituting the values of dl and de from (324) and (326), together with $\sin^2\varphi = \frac{1}{3}$

$$d\sigma = \sigma \left\{ \frac{1-\varepsilon}{3-e^2} - \frac{0.540\,94}{l(1-\frac{1}{2}\varepsilon)^2} \right\} d\varepsilon \tag{331}$$

Also, with sufficient accuracy

$$\frac{1+\sigma\sqrt{\frac{2}{3}}}{1+(\sigma+d\sigma)\sqrt{\frac{2}{3}}} = 1 - \frac{d\sigma\sqrt{\frac{2}{3}}}{1+\sigma\sqrt{\frac{2}{3}}}$$

and therefore

$$\log\frac{1+\sigma\sqrt{\frac{2}{3}}}{1+(\sigma+d\sigma)\sqrt{\frac{2}{3}}} = -\frac{\sigma M\sqrt{\frac{2}{3}}}{1+\sigma\sqrt{\frac{2}{3}}} \left\{ \frac{1-\varepsilon}{3-e^2} - \frac{0.540\,94}{l(1-\frac{1}{2}\varepsilon)^2} \right\} d\varepsilon \tag{332}$$

With $M = 0.434\,294$, and $\varepsilon = 1/293.466$, $e^2 = 0.006\,803\,48$, $l = 3.256\,87$ feet, $\sigma = 0.002\,829\,7$, $1+\sigma\sqrt{\frac{2}{3}} = 1.002\,31$, from (4), (5), (14), and (154), (325), (329), and (332) give

$$\tfrac{1}{3}\log\frac{l}{l+dl} + \tfrac{1}{3}\log\frac{\rho^2}{(\rho+d\rho)^2} + \tfrac{1}{3}\log\frac{1+\sigma\sqrt{\frac{2}{3}}}{1+(\sigma+d\sigma)\sqrt{\frac{2}{3}}}$$
$$= \tfrac{1}{3}(-0.072\,379 + 0.286\,564 - 0.000\,166)d\varepsilon \tag{333}$$

A comparison of (152) with (157) shows that the logarithmic coefficient of the latter is

$$\log\frac{4a(1+\varkappa')^3}{l\rho^2 T_1^2(1+\sigma\sqrt{\frac{2}{3}})} = 4.665\,070\,71 - 10$$

whence, by reasoning identical with that employed in connection with (319)

$$\log\frac{4a(1+\varkappa')^3}{(l+dl)(\rho+d\rho)^2 T_1^2[1+(\sigma+d\sigma)\sqrt{\frac{2}{3}}]} = 4.665\,070\,71 + 0.214\,02\,d\varepsilon - 10 \tag{334}$$

Upon equating (157) and (165) we obtain an expression of the form

$$\mathfrak{A} = \mathfrak{L}\left\{\frac{1-\frac{1}{A}\sin^3 P}{\frac{1}{AB}+\frac{C}{B}-\frac{C}{AB}\sin^3 P}\right\} \tag{335}$$

where, from (334) and (165)

$1/A = 216\,236.65 - 106\,561\,d\varepsilon \qquad B = 10\,288\,642 \qquad C = 38\,442\,769$

and the substitution of these values gives the third formula of (336).

Collecting our results, we have from (156), (157), (166), (320), (333), (334), and (335)

$$p = [2.784\,993\,22 + 0.071\,34\,d\varepsilon]\left(\frac{E}{1+M}\right)^{\frac{1}{3}}$$
$$\frac{E'}{M} = \frac{\sin^3 P}{[4.665\,070\,71 + 0.214\,02\,d\varepsilon - 10] - \sin^3 P} \tag{336}$$
$$\mathfrak{A} = \mathfrak{L}\left\{\frac{1-(216\,236.65 - 106\,561\,d\varepsilon)\sin^3 P}{3.757\,444\,9 - 0.010\,357\,d\varepsilon - (807\,952.64 - 398\,157\,d\varepsilon)\sin^3 P}\right\}$$

These formulæ are adapted to any value of the flattening between the limits 1 : 290 and 1 : 300, and in order to facilitate their use the following supplementary expressions are added:

$$\begin{aligned} d\varepsilon &= \varepsilon - 1/293{\cdot}466 = \varepsilon - 0{\cdot}003\,407\,55 \\ \log\,(216\,236{\cdot}648 - 106\,561\,d\varepsilon) &= 5{\cdot}334\,929\,300 - 0{\cdot}214\,020\,d\varepsilon \\ \log\,(3{\cdot}757\,444\,92 - 0{\cdot}010\,357\,d\varepsilon) &= 0{\cdot}574\,892\,623 - 0{\cdot}001\,197\,d\varepsilon \\ \log\,(807\,952{\cdot}643 - 398\,157\,d\varepsilon) &= 5{\cdot}907\,385\,906 - 0{\cdot}214\,021\,d\varepsilon \end{aligned} \qquad (337)$$

With a flattening of 1 : 300, $d\varepsilon = -0{\cdot}000\,074\,22$, and the conditions which must be satisfied by the adjusted quantities are, from (336), (175), (180), (182), and (18

$$\begin{aligned} \frac{1}{M} &= \frac{\sin^3 P}{[4{\cdot}665\,054\,83 - 10] - \sin^3 P} \\ 0 \text{ or } v_1 &= p - [2{\cdot}784\,988]\left(\frac{E}{1+M}\right)^{\frac{1}{3}} \\ 0 \text{ or } v_2 &= p - [5{\cdot}303\,125 - 10]\,PQ\,\frac{1+M}{1-M} \\ 0 \text{ or } v_3 &= p - [4{\cdot}681\,962 - 10]\,PL\,\frac{1+M}{M} \\ 0 \text{ or } v_4 &= p - \frac{[8{\cdot}912\,482]}{V\theta} \\ 0 \text{ or } v_5 &= p - \frac{[7{\cdot}526\,036]}{V\alpha} \\ 0 \text{ or } v_6 &= \mathfrak{N} - \mathfrak{L}\left\{\frac{1 - 216\,244{\cdot}56 \sin^3 P}{3{\cdot}757\,445\,7 - 807\,982{\cdot}20 \sin^3 P}\right\} \end{aligned} \qquad (338)$$

The values of the observed quantities will be the same as in (195); except those of P and $\mathfrak{L}$, for which (70), (75), and (204) give

$$\begin{aligned} P &= 3\,423{\cdot}08'' + 5\,062''\,d\varepsilon \pm 0{\cdot}121'' \\ \mathfrak{L} &= 50{\cdot}358\,6'' - 31\,716''\,dE \pm 0{\cdot}002\,48'' \end{aligned} \qquad (339)$$

With the above value of $d\varepsilon$, and $dE = +0{\cdot}000\,000\,051$, we therefore have

$$\begin{aligned} p &= 8{\cdot}834'' \pm 0{\cdot}008\,6'' \\ P &= 3\,422{\cdot}704 \pm 0\,121 \\ \mathfrak{L} &= 50{\cdot}357\,0 \pm 0{\cdot}002\,48 \\ \mathfrak{N} &= 9{\cdot}233\,1 \pm 0{\cdot}011\,2 \\ Q &= 125{\cdot}46 \pm 0{\cdot}35 \\ L &= 6{\cdot}514 \pm 0{\cdot}016 \\ \alpha &= 20{\cdot}466 \pm 0{\cdot}011 \\ \theta &= 497{\cdot}0^{s} \pm 1{\cdot}02^{s} \\ V &= 186\,347 \pm 36 \text{ miles} \\ E &= 0{\cdot}000\,003\,005\,097 \pm 0{\cdot}000\,000\,016\,056 \end{aligned} \qquad (340)$$

The substitution of these observed values in the conditional equations (338) leads to the following system of numbers

$$1/M = 81\cdot722\,6$$

$$\begin{aligned} v_1 &= +0\cdot073\,81'' & v_4 &= +0\cdot007\,20'' \\ v_2 &= -0\cdot009\,59 & v_5 &= +0\cdot029\,99 \\ v_3 &= -0\cdot033\,55 & v_6 &= +0\cdot035\,60 \end{aligned} \qquad (341)$$

from which the corrections by adjustment given in Table XIII result by a double application of the formulæ (193), precisely as in the case of Table VI.

It is now desirable to have a method more direct than that employed on page 72 for finding the probable errors of the masses of Mercury and Venus after they have been corrected on account of dE. As the expressions (203) are of the form $A' + B'dE = (1 + \nu)/m_0$, the expressions for ν, ν^{i}, and ν^{ii} will be of the form $\nu = A'' + B''dE$, and when they are substituted in the observation equations (129) the resulting residuals will be of the form, $v = A''' + B'''dE$. We may therefore write

$$\Sigma vv = A + B.dE + C(dE)^2 \qquad (342)$$

and if the probable error of dE is $\pm\, dE'$, when that quantity is given there will be an additional term of the form $+ D(dE')^2$. The algebraic expressions for A, B, C and D are simple enough, but they are not needed here because the numerical values of these constants can be most readily found by an indirect process.

The residuals on page 48 give $\Sigma vv = 4\cdot257\,31$; those on page 72 give, for $dE = +\,51\,440 \pm 27\,331$, $\Sigma vv = 5\cdot500\,65 + 0\cdot578\,55$, where $0\cdot578\,55$ is the part arising from the probable error of dE; and a special computation gives, for $dE = +\,25\,000$, $\Sigma vv = 4\cdot551\,04$. In accordance with (342), these numbers yield the equations

$$\begin{aligned} 4\cdot257\,31 &= A \\ 5\cdot500\,65 &= A + 5\cdot144\,0\,B + 26\cdot460\,74\,C \\ 4\cdot551\,04 &= A + 2\cdot500\,0\,B + 6\cdot250\,00\,C \\ 0\cdot578\,55 &= 7\cdot469\,84\,D \end{aligned} \qquad (343)$$

where dE and dE' have been multiplied by 100 000 000 for convenience in printing. The solution of (343) gives

$$\begin{aligned} \Sigma vv = 4\cdot257\,31 + [3\cdot627\,3]dE + [14\cdot671\,91](dE)^2 \\ + [14\cdot889\,03](dE')^2 \end{aligned} \qquad (344)$$

the quantities within brackets being the logarithms of the numbers they represent.

From (198), (203), and (344), together with the usual formula for probable error, the following general expressions for the corrected masses of Mercury, Venus and the Earth have been derived:

$$\begin{aligned} \text{Mass of Mercury} &= \frac{0\cdot344\,647 + 266\,200\,dE \pm R}{3\,000\,000} \\ \text{Mass of Venus} &= \frac{0\cdot992\,996 - 204\,107\,dE \pm R'}{401\,847} \\ \text{Mass of Earth} &= \frac{1\cdot066\,618 + 354\,936\,(dE \pm dE')}{354\,936} \end{aligned} \qquad (345)$$

where dE is the correction to the mass of the Earth given by (193), dE′ its probable error, and

$$\left.\begin{array}{l}\text{Log R} = \frac{1}{2}\{\log. \Sigma vv + 6{\cdot}979\,147\} \\ \text{Log R}' = \frac{1}{2}\{\log. \Sigma vv + 4{\cdot}566\,211\}\end{array}\right\} \qquad (346)$$

The probable errors in the second and fourth columns of Table XIII have been computed from the data in Table V, with $q = 1{\cdot}375\,8$; and the values of the appended planetary masses, together with their probable errors, have been computed by means of the formulæ (344) to (346), thus avoiding the laborious process described on page 72.

TABLE XIII.—*Results for the Epoch* 1850·0, *upon the assumption that the Earth's Flattening is* 1 : 300·00.

Quantities.	Observed values.	Corrections by adjustment.	Adjusted values.
	″ ″	″	″ ″
p	8·834 ± 0·011 83	− 0·024 63	8·809 37 ± 0·005 54
P	3 422·704 ± 0·166 47	− 0·107 74	3 422·596 26 ± 0·038 97
𝔓	50·357 0 ± 0·003 41	+ 0·000 12	50·357 12 ± 0·003 41
𝔑	9·233 1 ± 0·015 41	− 0·013 16	9·219 94 ± 0·008 23
Q	125·46 ± 0·481 53	− 0·505 38	124·954 62 ± 0·080 03
L	6·514 ± 0·022 01	+ 0·007 40	6·521 40 ± 0·018 04
a	20·466 ± 0·015 13	− 0·011 35	20·454 65 ± 0·012 36
θ	497·0ˢ ± 1·403 32ˢ	+ 1·009 81ˢ	498·009 81ˢ ± 0·300 67ˢ
V	186 347 ± 49·529 miles	−10·009 miles	186 336·99 ± 48·542 miles
E	0·000 003 005 097 ± 0·000 000 022 090	+51·1840	0·000 003 056 281 ± 0·000 000 005 784
M	. . . .	. . . .	0·012 332 07 ± 0·000 034 58

$$\text{Mass of Mercury} = \frac{0{\cdot}358\,272 \pm 0{\cdot}072\,496}{3\,000\,000} = \frac{1}{8\,373\,526 \pm 1\,766\,213}$$

$$\text{Mass of Venus} = \frac{0{\cdot}982\,549 \pm 0{\cdot}004\,507}{401\,847} = \frac{1}{408\,984 \pm 1\,876}$$

$$\text{Mass of Earth} = \frac{1{\cdot}084\,785 \pm 0{\cdot}002\,053}{354\,936} = \frac{1}{327\,195 \pm 619}$$

$$\text{Mass of Moon} = \frac{1}{81{\cdot}089\,4 \pm 0{\cdot}227\,4}$$

Mean distance from Earth to Sun = 92 797 600 ± 58 318 miles.
Mean distance from Earth to Moon = 238 861 ± 2·719 miles.

As the correction which would accrue to the equatorial diameter of the Earth upon changing the flattening from 1 : 293·5 to 1 : 300 is of the same order as the probable uncertainty in the diameter itself, the distances of the Sun and Moon appended to Table XIII have been computed with the same equatorial radius as those appended to Table VI. Nevertheless, it is important to inquire how a small change in the flattening will affect our adopted values of the Earth's equatorial and polar semidiameters, and we therefore revert to General CLARKE's final equations for the determination of these quantities. They are*

$$\begin{aligned} 0 &= +56.661\,5 + 301.762\,4\,u + 126.925\,2\,v \\ 0 &= -16.967\,7 + 126.925\,2\,u + 221.430\,7\,v \end{aligned} \tag{347}$$

and our first task will be to express u and v in terms of db and $d\varepsilon$. To that end, let

$$\begin{aligned} a &= \frac{b}{1-\varepsilon} = a_0 + da & \qquad b &= b_0 + db \\ \varepsilon &= \varepsilon_0 + d\varepsilon & \qquad b_0 &= 20\,855\,500 \text{ feet} \end{aligned} \tag{348}$$

Then†

$$b = b_0 + db = b_0 \left\{ 1 + \frac{u}{10\,000} \right\} \tag{349}$$

$$n = \frac{a-b}{a+b} = \frac{\varepsilon}{2-\varepsilon} = \frac{1}{590} + 10v \text{ arc } 1'' \tag{350}$$

and from (349)

$$u = 0.000\,479\,490\,db \tag{351}$$

Further, as

$$\frac{\varepsilon}{2-\varepsilon} = \frac{\varepsilon_0 + d\varepsilon}{2 - \varepsilon_0 - d\varepsilon} = \frac{\varepsilon_0}{2-\varepsilon_0} + \left\{ \frac{1}{2-\varepsilon_0} + \frac{\varepsilon_0}{(2-\varepsilon_0)^2} \right\} d\varepsilon \tag{352}$$

we may write $\varepsilon_0/(2-\varepsilon_0) = 1/590$, and from that, together with (350) and (352)

$$\begin{aligned} &\left\{ \frac{1}{2-\varepsilon_0} + \frac{\varepsilon_0}{(2-\varepsilon_0)^2} \right\} d\varepsilon = 10v \text{ arc } 1'' \\ &\varepsilon_0 = 1/295.5 = 0.003\,384\,095 \qquad v = 10\,348.2\,d\varepsilon \end{aligned} \tag{353}$$

The substitution of the values of u and v from (351) and (353) in the first equation of (347) gives

$$db = -391.6 - 9\,077\,539\,d\varepsilon \tag{354}$$

and by differentiating the expression $a_0(1-\varepsilon_0) = b_0$, and substituting the values of b_0, ε_0, and db from (348), (353), and (354)

$$da = \frac{db}{1-\varepsilon_0} + \frac{b_0 d\varepsilon}{(1-\varepsilon_0)^2} = -392.9 + 11\,889\,011\,d\varepsilon \tag{355}$$

Then (348), (353), (354), and (355) give

$$\begin{aligned} a &= b_0/(1-\varepsilon_0) + da = 20\,925\,923.7 + 11\,889\,011\,d\varepsilon \text{ feet} \\ b &= b_0 + db \qquad\qquad\quad\;\; = 20\,855\,108.4 - 9\,077\,539\,d\varepsilon \text{ feet} \end{aligned}$$

and by changing ε_0 from $1/295.5$ to $1/293.47$, we obtain finally

$$\begin{aligned} a &= 20\,926\,202 + 11\,889\,011\,d\varepsilon \text{ feet} \\ b &= 20\,854\,895 - 9\,077\,539\,d\varepsilon \text{ feet} \end{aligned} \tag{356}$$

* 13, p. 317.

† 13, pp. 111 and 313.

For a flattening of 1 : 300, (337) gives $d\varepsilon = -0.000\,074\,22$, and by (356) the corresponding equatorial and polar semidiameters of the Earth are

$$\begin{aligned} a &= 20\,925\,320 \text{ feet} \\ b &= 20\,855\,569 \text{ feet} \end{aligned} \tag{357}$$

Upon combining this new value of a with the adjusted parallaxes contained in Table XIII we find

$$\begin{aligned} &\text{Mean distance from Earth to Sun} = 92\,793\,663 \pm 58\,318 \text{ miles} \\ &\text{Mean distance from Earth to Moon} = 238\,851 \pm 2.719 \text{ miles} \end{aligned} \tag{358}$$

and these results are substantially the same as those appended to Table XIII, the discordance amounting to ten miles in the case of the Moon, and to only one-fifteenth of the probable error in the case of the Sun.

The most striking feature elicited by comparing Tables VI and XIII is the great diminution of the probable errors in the latter; and as that arises solely from the change in the flattening, and extends not only to the quantities in the table, but also to the appended masses derived from the equations (129), it seems desirable to inquire what value of the flattening would reduce these probable errors to a minimum. A rigorous solution of that question would be very laborious, but a fair approximation may be got by finding what value of ε would make the residuals in (185) a minimum. If v' and v'' are the two values of any one of these residuals, given respectively in (196), (206), and (341), and corresponding to the flattenings ε' and ε'', then the condition that that particular residual shall disappear may be written

$$0 = v' + \frac{v' - v''}{\varepsilon' - \varepsilon''} d\varepsilon \tag{359}$$

and the resulting value of the flattening will be $\varepsilon' + d\varepsilon$. The residuals v_4 and v_5 are independent of the flattening, but v_1, v_2, v_3, and v_6 furnish the four observation equations

$$\begin{aligned} 0 &= +0.072\,86'' - 12.8''\, d\varepsilon \\ 0 &= -0.005\,32 + 57.5\, d\varepsilon \\ 0 &= -0.252\,22 - 2\,946.6\, d\varepsilon \\ 0 &= +0.106\,66 + 957.5\, d\varepsilon \end{aligned}$$

from which we find, by the method of least squares

$$\varepsilon' + d\varepsilon = \frac{1}{293.466} - 0.000\,087\,90 = \frac{1}{301.24} \tag{360}$$

It is not easy to decide how much weight should be attributed to this method of estimating ε, but the result certainly tends to strengthen HELMERT's value from pendulum experiments, and our own from precession and nutation.

Doubtless more data are the first requisite for a satisfactory determination of the Earth's flattening, but in default thereof one resource yet remains, and we shall avail ourselves of it in Section 27. Meanwhile a symbolic correction may be applied to General CLARKE's geodetic value, by writing

$$\varepsilon = 1/293.466 + d\varepsilon \tag{361}$$

and then every quantity in, and appended to, Table VI will take the form

$$x = x' + \frac{x' - x''}{\varepsilon' - \varepsilon''}\, d\varepsilon \tag{362}$$

where x is any one of the quantities in question, x' and x'' are respectively the numerical values given for it in Tables VI and XIII, and ε' and ε'' are the values of the flattening with which these tables were computed. As $\varepsilon' = 1/293.466\,3$, and $\varepsilon'' = 1/300$, equation (362) reduces to

$$x = x' + 13\,475\,(x' - x'')\, d\varepsilon \tag{363}$$

The resulting numerical expressions are appended, and by their aid the values of all the quantities in question can readily be found for any possible value of the flattening.

$$d\varepsilon = \varepsilon - 1/293.466\,3 = \varepsilon - 0.003\,407\,546$$

$$p = 8.809\,76'' + 5.26''\,d\varepsilon \pm (0.006\,74'' + 16.17''\,d\varepsilon)$$

$$\mathrm{P} = 3\,422.656\,43'' + 810.79''\,d\varepsilon \pm (0.047\,46'' + 114.40''\,d\varepsilon)$$

$$\mathfrak{P} = 50.357\,15'' + 0.40''\,d\varepsilon \pm (0.004\,15'' + 9.97''\,d\varepsilon)$$

$$\mathfrak{N} = 9.216\,21'' - 50.26''\,d\varepsilon \pm (0.010\,02'' + 24.12''\,d\varepsilon)$$

$$\mathrm{Q} = 124.962\,05'' + 100.12''\,d\varepsilon \pm (0.097\,46'' + 234.87''\,d\varepsilon)$$

$$\mathrm{L} = 6.513\,03'' - 112.79''\,d\varepsilon \pm (0.021\,97'' + 52.96''\,d\varepsilon)$$

$$\alpha = 20.453\,85'' - 10.78''\,d\varepsilon \pm (0.015\,06'' + 36.38''\,d\varepsilon)$$

$$\theta = 497.990\,94^{s} - 254.27^{s}\,d\varepsilon \pm (0.366\,16^{s} + 882.48^{s}\,d\varepsilon)$$

$$\mathrm{V} = 186\,335.99 - 13\,475\,d\varepsilon \pm (59.117 + 142\,498\,d\varepsilon)$$

Mass of Mercury =

$$\left.\begin{matrix} 0.000\,000\,119\,477 \\ +\,0.000\,000\,310\,d\varepsilon \end{matrix}\right\} \pm \left\{\begin{matrix} 0.000\,000\,024\,220 \\ +\,0.000\,000\,732\,d\varepsilon \end{matrix}\right. \tag{364}$$

Mass of Venus =

$$\left.\begin{matrix} 0.000\,002\,444\,953 \\ -\,0.000\,001\,738\,d\varepsilon \end{matrix}\right\} \pm \left\{\begin{matrix} 0.000\,000\,011\,241 \\ +\,0.000\,000\,335\,d\varepsilon \end{matrix}\right.$$

Mass of Earth =

$$\left.\begin{matrix} 0.000\,003\,056\,537 \\ +\,0.000\,003\,423\,d\varepsilon \end{matrix}\right\} \pm \left\{\begin{matrix} 0.000\,000\,007\,043 \\ +\,0.000\,016\,965\,d\varepsilon \end{matrix}\right.$$

Mass of Moon =

$$0.012\,315\,7 - 0.221\,0\,d\varepsilon \pm (0.000\,042\,11 + 0.101\,5\,d\varepsilon)$$

Mean distance from Earth to Sun =

$$92\,793\,504 - 54\,857\,000\,d\varepsilon \pm (70\,993 + 170\,796\,000\,d\varepsilon) \text{ miles}$$

Mean distance from Earth to Moon =

$$238\,857 - 53\,900\,d\varepsilon \pm (3.312 + 7\,991\,d\varepsilon) \text{ miles.}$$

26.—MASS OF THE MOON FROM OBSERVATIONS OF THE TIDES.

The first determination of the Moon's mass was made by NEWTON in 1687, from the tides, and other investigators have since employed the same method, but for more than 180 years it yielded no trustworthy result. Its failure was due to various causes, both theoretical and practical, and although some of these were cleared up by LA PLACE as early as 1818, there was little prospect of success until the recent application of harmonic analysis to the reduction of continuous observations of the tides, recorded by automatic gauges, and extending over long periods of time. An effort has been made to collect in the following list all the more important determinations of the Moon's mass which are based upon the method in question, but owing to the singular sparseness of bibliographic references in writings on that subject, it is difficult to estimate what degree of completeness has been attained.

Reciprocals of the Moon's Mass, determined from Observations of the Tides.

1687. NEWTON, from the tides before the mouth of the river Avon, three miles below Bristol, England. (9, lib. 3, prop. 37, cor. 4.)	39·8
1738. D$_1$. BERNOULLI. (5, p. 549.)	70·
1818. LA PLACE, from the tides at Brest, France, observed during the years 1807 to 1814, inclusive. (348, p. 55.)	69·3
1824. LA PLACE, from a rediscussion of the above observations at Brest. (7, T. 5, liv. 13, chap. 3, § 10.)	74·9
1831. LUBBOCK, from the tides at London, England, observed during the years 1808 to 1826, inclusive. (350, 1831, p. 392.)	66·7
1854. HAUGHTON, from the semi-diurnal tides on the coast of Ireland, observed during the year 1851. (344, p. 130.)	63·
From the diurnal tides, during the same period. (344, p. 130.)	95·
1860. LUBBOCK remarks in his paper on the lunar theory (Mem. Roy. Ast. Soc. 1861, vol. 30, p. 29) that the observations of the tides which he employed gave him a value "probably about 1:67·3" for the ratio of the mass of the Moon to the sum of the masses of the Earth and Moon. His value of the reciprocal of the Moon's mass must therefore have been 66·3, which agrees with what he found from the tides at London, in 1831, so closely as to render it probable that they are the tides referred to.	
1862. HAUGHTON, from the semi-diurnal tides at Port Leopold, North Somerset, observed November, 1848, to July, 1849, inclusive. (345, 1866, p. 655.)	65·4
From the diurnal tides during the same period. (345, 1863, p. 253.)	85·
1866. HAUGHTON, from the semi-diurnal tides at Frederiksdal, Greenland, observed August, 1863, to August, 1864, inclusive. (345, 1866, p. 642.)	64·6
1866. HAUGHTON, from the semi-diurnal tides, observed at the following points on the coast of Ireland, during the year 1851. (346, p. 346.):	
Bunown	69·1
Cahirciveen	64·2
Castletownsend	49·2
Dunmore East	55·0
Courtown	107·2
Kingstown	46·3
Donaghadee	35·1
Cushendall	21·1
Portrush	84·6
Rathmullan	71·7
Mean	60·4
1867. FINLAYSON, from the mean range of the spring and neap tides at Dover, England, observed during the years 1861, 1864, 1865, and 1866. (343, p. 272.)	87·9
1870. FERREL, from observations of the tides at Boston, Mass., from July 1, 1847, to July 1, 1866. (334, p. 85.)	78·6
1871. FERREL, from a rediscussion of the tides at Boston, Mass., July 1, 1847, to July 1, 1866, with special reference to the Moon's mass. (335, p. 198.)	75·1
1874. FERREL, from observations of the tides at Brest, France, during the years 1812 to 1832, inclusive. (336, p. 189.)	78·0
1874. FERREL, from a second revision of his discussion of the tides observed at Boston, Mass., July 1, 1847, to July 1, 1866. (336, p. 196.)	81·7

1874. FERREL, from seven years' observations of the tides at Liverpool, England, 1857 to 1860, and 1866 to 1870. (336, p. 216.)	73·3
1874. FERREL, from observations of the tides at Portland Breakwater, England, during the years 1851, 1857, 1860, and and 1870. (336, p. 223.)	80·1
1874. FERREL, from observations of the tides at Fort Point, Cal., during the three years 1858 to 1861. (336, p. 228.) .	76·6
1874. FERREL, from observations of the tides at Karachi, India, during the three years 1868 to 1871. (336, p. 234.) .	78·6
1878. FERREL, from observations of the tides made at Pulpit Cove, Penobscot Bay, Maine, during the six years 1870 to 1875. (338, p. 294.)	83·3
1882. FERREL, from observations of the tides made at Port Townsend, Wash., during the three years 1874 to 1876. (339, p. 448.)	77·2
1882. FERREL, from observations of the tides made at Astoria, Oregon, during the three years 1874 to 1876. (339, p. 448.)	68·1
1882. FERREL, from observations of the tides made at San Diego, Cal., during the three years 1869 to 1871. (339, p. 448.)	88·0
1883. FERREL, from observations of the tides made at Sandy Hook, N. J., during the six years 1876 to 1881, inclusive. (340. p. 251.)	77·1

Long ago AIRY showed why the Moon's mass can not be accurately determined from the mere ratio of the solar and lunar effects in the semi-mensual inequality of the tides,* but nevertheless many of the values recorded above have been obtained in that very way, and are therefore worthless. Those found by LA PLACE's method,† or by FERREL's modification of it, are theoretically correct, at least for deep-water tides, but instead of confining ourselves to them, we shall compute many new values from the "Results of the harmonic analysis of tidal observations" which have been published by Major BAIRD and Professor DARWIN.‡

LA PLACE's method of deducing the Moon's mass is not adapted for use with harmonic tidal constants. We shall therefore employ FERREL's formulæ, and as much of the available data was collected by Sir WILLIAM THOMSON, Mr. EDWARD ROBERTS, Major A. W. BAIRD, R. E. and Professor G. H. DARWIN, the various notations adopted by these gentlemen are exhibited in Table XIV. FERREL used the letters A or a for the semi-range, and ε or e for the epoch of a tide, and distinguished the various classes of components by the subscript suffixes 1, 2, 3, etc.; but for similar components occurring in the diurnal and semi-diurnal tides he used the same symbols. THOMSON used the letters R and ϵ, respectively, for the semi-range and epoch of any component, distinguishing the various classes of components by the initials S, M, O, K, etc., and indicating their period by the subscript suffixes 1, 2, 3, etc.; 1 indicating a diurnal component, 2 a semi-diurnal component, 3 a terdiurnal component, and so on. DARWIN used the same initials as THOMSON to designate the various classes of tidal components, but his H and $\varkappa$ are not identical with FERREL's A and ε, and THOMSON's R and ϵ. The semi-ranges and epochs of most of the tidal components are to a certain extent functions of the longitude of the Moon's node, and are therefore subject to small inequalities having a period of 18.6 years. The A's and ε's of FERREL and the R's and ϵ's of THOMSON are affected by these inequalities, but the H's and $\varkappa$'s of DARWIN are very nearly free from them.§ In other words, the A's and ε's of FERREL and the R's and ϵ's of THOMSON belong strictly to the years during which they were observed, but the H's and $\varkappa$'s of DARWIN are reduced to what they would have been if the Moon

* 323, p. 360*, art. 455.

† The basis of this method is very clearly explained by AIRY, 323, pp. 360*, 379*, and 386*, articles 455, 536, and 555.

‡ 327.

§ Compare 353, 1883, p. 86, and 338, p. 282.

had remained always in the celestial equator, and they therefore correspond to the means of a series of observations extending over the entire period of 18.6 years, except in so far as they may be vitiated by accidental errors. In explanation of Table XIV it is only necessary to add that quantities on the same line have similar significations. For example, FERREL's diurnal A_1 and ε_1 correspond to THOMSON's R_1 and ϵ_1 of K_1, and to DARWIN's H and $\varkappa$ of K_1; and similarly for the quantities on the other lines.

TABLE XIV.—*Notation for Harmonic Analysis of the Tides.*

	FERREL.	THOMSON and ROBERTS.	DARWIN and BAIRD.	Name of Tide.	Speed of Tide per Mean Solar Hour.
	H_0	A_0	. . .	Mean level of sea.	°
Diurnal components.	A_1 ε_1	K_1 , R_1 ϵ_1	K_1 , H κ	Luni-solar diurnal	15·041 068 6
	A_2 ε_2	O , R_1 ϵ_1	O , H κ	Lunar diurnal	13·943 035 6
	A_3 ε_3	P , R_1 ϵ_1	P , H κ	Solar diurnal	14·958 931 4
	a_0 e_0	S_1 , R_1 ϵ_1	S_1 , H κ	1st of principal solar series.	15·000 000 0
Semi-diurnal components.	A_0 ε_0	M_2 , R_2 ϵ_2	M_2 , H κ	2nd of principal lunar series .	28·984 104 2
	A_1 ε_1	S_2 , R_2 ϵ_2	S_2 , H κ	2nd of principal solar series .	30·000 000 0
	A_2 ε_2	μ , R_2 ϵ_2	μ , H κ	Compound tide	27·968 208 4
	A_3 ε_3	K_2 , R_2 ϵ_2	K_2 , H κ	Luni-solar semi-diurnal . .	30·082 137 2
	A_4 ε_4	L , R_2 ϵ_2	L , H κ	Smaller lunar elliptic . . .	29·528 478 8
	A_5 ε_5	N , R_2 ϵ_2	N , H κ	Larger lunar elliptic	28·439 729 6
Shallow-water components.	a_1 e_1	M_4 , R_4 ϵ_4	M_4 , H κ	4th of principal lunar series .	57·968 208 4
	a_2 e_2	MS , R_4 ϵ_4	MS , H κ	Compound tide	58·984 104 2
	a_3 e_3	S_4 , R_4 ϵ_4	S_4 , H κ	4th of principal solar series .	60·000 000 0
	a_4 e_4	2SM , R_2 ϵ_2	2SM , H κ	Compound tide	31·015 895 8
	a_5 e_5	M_6 , R_6 ϵ_6	M_6 , H κ	6th of principal lunar series .	86·952 312 6
	a_6 e_6	. . .	. . .		

Our object is to determine the mass of the Moon, and as that quantity affects only the ranges of the tidal components, and is without influence on their epochs, we shall have to deal exclusively with the H's, and not at all with the $\varkappa$'s. Our notation may therefore be abridged by using the initials of the various tides to denote their semi-ranges, and instead of H of S, H of M, H of O, etc., we shall write simply S, M, O, etc.

For the diurnal tidal oscillations of the great oceans, Professor FERREL's expressions are*

$$
\begin{aligned}
K_1 &= A_1 = (0{\cdot}530\,6 - 13{\cdot}1\,\delta\mu)(1 + 0{\cdot}230\,E_1)\,A_0 \\
O &= A_2 = 0{\cdot}381\,3\,(1 - 0{\cdot}230\,E_1)\,A_0 \\
P &= A_3 = (0{\cdot}173\,0 - 13{\cdot}6\,\delta\mu)(1 + 0{\cdot}196\,E_1)\,A_0 \\
& \quad A_4 = 0{\cdot}084\,(1 + 0{\cdot}231\,E_1)\,A_0 \\
& \quad A_5 = 0{\cdot}070\,(1 - 0{\cdot}231\,E_1)\,A_0
\end{aligned}
\qquad (365)
$$

*336, p. 89.

where A_0, E_1, and $\delta\mu$ are constants to be determined from the values of A_1, A_2, A_3, etc., by the simultaneous solution of three or more of the equations. As the numerical values of A_4 and A_5 are not contained in BAIRD and DARWIN's list of harmonic tidal constants, we shall neglect the last two equations of (365); but their loss is of little consequence on account of their small weight. By putting

$$\begin{aligned} m &= 13{\cdot}6\ K_1 - 12{\cdot}98\ O - 13{\cdot}1\ P \\ 2a &= 0{\cdot}462\ 4\ K_1 + 5{\cdot}529\ O \\ b &= 0{\cdot}613\ 1\ K_1 + 0{\cdot}585\ 1\ O - 0{\cdot}693\ 0\ P \end{aligned}$$

the remaining equations of (365) give

$$\begin{aligned} E_1 &= -\frac{a}{b} \pm \left\{ \frac{m}{b} + \frac{a^2}{b^2} \right\}^{\frac{1}{2}} \\ A_0 &= \frac{O}{0{\cdot}381\ 3\,(1 - 0{\cdot}230\ E_1)} \\ \delta\mu &= 0{\cdot}040\ 50 - \frac{0{\cdot}381\ 3\,(1 - 0{\cdot}230\ E_1)\ K_1}{13{\cdot}1\,(1 + 0{\cdot}230\ E_1)\ O} \\ \text{Mass of Moon} &= 0{\cdot}012\ 50 + \delta\mu \end{aligned} \tag{366}$$

Although this solution is in the form best adapted to give exact numerical results, it is too cumberous for general use, and we take advantage of the smallness of $\delta\mu$ to add another which is briefer and not appreciably less accurate:

$$\begin{aligned} \delta\mu &= \frac{0{\cdot}122\ 30\ K_1 + 0{\cdot}013\ 58\ O - 0{\cdot}405\ 04\ P}{9{\cdot}614\ 36\ K_1 + 1{\cdot}513\ 63\ O - 10{\cdot}000\ 00\ P} \\ E_1 &= \frac{K_1 - O\,(1{\cdot}391\ 56 - 34{\cdot}36\ \delta\mu)}{0{\cdot}230\ K_1 + O\,(0{\cdot}320\ 06 - 7{\cdot}90\ \delta\mu)} = \frac{P - O\,(0{\cdot}453\ 71 - 35{\cdot}67\ \delta\mu)}{0{\cdot}230\ P + O\,(0{\cdot}088\ 93 - 6{\cdot}99\ \delta\mu)} \\ A_0 &= \frac{O}{0{\cdot}381\ 3\,(1 - 0{\cdot}230\ E_1)} \\ \text{Mass of Moon} &= 0{\cdot}012\ 50 + \delta\mu \end{aligned} \tag{367}$$

For the semi-diurnal tidal oscillations of the great oceans, Professor FERREL gives the expressions*

$$\begin{aligned} S_2/M_2 = R_1 &= (0{\cdot}458\ 2 - 36{\cdot}2\delta\mu)(1 + 0{\cdot}425\ 5\ E_2) \\ \mu/M_2 = R_2 &= +0{\cdot}024\ 0\,(1 - 0{\cdot}425\ 5\ E_2) \\ K_2/M_2 = R_3 &= (0{\cdot}125\ 6 - 3{\cdot}2\delta\mu)(1 + 0{\cdot}459\ 9\ E_2) \\ L/M_2 = R_4 &= -0{\cdot}028\ 6\,(1 + 0{\cdot}228\ E_2) \\ N/M_2 = R_5 &= +0{\cdot}192\ 2\,(1 - 0{\cdot}228\ E_2) \end{aligned} \tag{368}$$

The second and fourth of these equations may be dismissed at once on account of their small weight, but there is a difficulty in deciding how the remaining three should be treated in order to get from them the most probable values of E_2 and $\delta\mu$. The fundamental quantities M_2, S_2, K_2, and N are not observed independently, but are each functions of the same observed quantities, namely, a series of heights recorded by a tide gauge, and therefore according to the theory of least squares it is the sum

*336, p. 91.

of the squares of the corrections to the latter which should be made a minimum, subject to the conditions of (368). But even if these heights were at hand the applicability of this process would be doubtful, because in the present case the larger errors probably do not follow the usual law of error. However, as the only practicable course is to deal with the derived functions M_2, S_2, etc., our discussion had better be confined to them. If, on the one hand, we take the equations as they stand, and solve by the method of least squares, the results are likely to be affected by systematic errors, because all the absolute terms have been divided by the same fundamental quantity, namely, M_2, which is itself subject to error; and if, on the other hand, we apply a symbolic correction to M_2, then we shall have as many unknown quantities as there are equations, and the errors which should be distributed among all the fundamental quantities will be thrown upon M_2 alone. Under these circumstances it is probably best to divide the third equation by the fifth, and thus we obtain the two equations

$$\begin{aligned} S_2/M_2 &= B = (0{\cdot}458\,2 - 36{\cdot}2\,\delta\mu)(1 + 0{\cdot}425\,5\,E_2) \\ K_2/N &= C = \frac{(0{\cdot}653\,5 - 16{\cdot}6\,\delta\mu)(1 + 0{\cdot}459\,9\,E_2)}{1 - 0{\cdot}228\,E_2} \end{aligned} \tag{369}$$

in which each of the fundamental quantities occurs once, and only once. Here, as in (365), we give two solutions. The first is that which leads to the most exact numerical results. For it we put

$$\begin{aligned} m &= -16{\cdot}051 - 16{\cdot}6\,B + 36{\cdot}2\,C \\ 2a &= +14{\cdot}211 + 7{\cdot}634\,3\,B - 7{\cdot}149\,5\,C \\ b &= +3{\cdot}140\,9 + 3{\cdot}511\,9\,C \end{aligned}$$

and then

$$\begin{aligned} E_2 &= -\frac{a}{b} \pm \left\{\frac{m}{b} + \frac{a^2}{b^2}\right\}^{\frac{1}{2}} \\ \delta\mu &= 0{\cdot}012\,66 - \frac{B}{36{\cdot}2 + 15{\cdot}403\,E_2} \\ \text{Mass of Moon} &= 0{\cdot}012\,50 + \delta\mu \end{aligned} \tag{370}$$

The second solution, which is briefer than the first and not appreciably less accurate, is as follows:

$$\begin{aligned} \delta\mu &= \frac{0{\cdot}096\,38\,BC + 0{\cdot}127\,04\,B - 0{\cdot}126\,57\,C - 0{\cdot}004\,354}{3{\cdot}236\,72\,B - 10{\cdot}000\,00\,C - 0{\cdot}454\,92} \\ E_2 &= \frac{C - 0{\cdot}653\,49 + 16{\cdot}65\,\delta\mu}{0{\cdot}228\,C + 0{\cdot}300\,54 - 7{\cdot}66\,\delta\mu} = \frac{B - 0{\cdot}458\,2 + 36{\cdot}2\,\delta\mu}{0{\cdot}194\,96 - 15{\cdot}40\,\delta\mu} \\ \text{Mass of Moon} &= 0{\cdot}012\,50 + \delta\mu \end{aligned} \tag{371}$$

Professor FERREL treats the equations (368) in a way which differs widely from that adopted above, and which is best explained by the following extract from his paper on the tides of Penobscot Bay:*

It is readily seen from an inspection of these equations that they can be satisfied only very imperfectly for Pulpit Cove, within any determined values of $\delta\mu$ and E, and that they can be

* 338, p. 297.

much better satisfied by multiplying the first members of the equations by an unknown constant. This constant is introduced upon the hypothesis that the tidal components are diminished by the effect of friction which is as a higher power than the first power of the velocity, as I have at various times explained. Upon this hypothesis large tides are diminished by friction more than small ones in proportion to their amplitudes, and hence where there is one large component, as the mean lunar, and a number of much smaller ones, since the amplitudes of the latter are obtained by analysis from the differences between the larger and smaller resultant tides, the smaller components are diminished more than the larger ones in proportion to their magnitudes, unless friction is as the first power of the velocity.

Accordingly, Professor FERREL multiplies the left-hand members of the first, third, and fifth equations of (368) by a constant which he calls c,* thus reducing them to the form

$$\begin{aligned} (S_2/M_2)\,c &= R_1 c = (0{\cdot}458\,2 - 36{\cdot}2\delta\mu)(1 + 0{\cdot}425\,5\,E_2) \\ (K_2/M_2)\,c &= R_3 c = (0{\cdot}125\,6 - 3{\cdot}2\delta\mu)(1 + 0{\cdot}459\,9\,E_2) \\ (N\,/M_2)\,c &= R_5 c = 0{\cdot}192\,2\,(1 - 0{\cdot}228E_2) \end{aligned} \qquad (372)$$

and he adds: "By the introduction of the constant c, or one equivalent to it, I have in all cases found that the observations are better represented by theory, and a better mass of the Moon is obtained, which indicates that there is an effect due to friction or some other cause which diminishes the amplitude of the tides."

If we put

$$\begin{aligned} m &= +\,3{\cdot}2\,R_1 - 36{\cdot}2\,R_3 + 16{\cdot}027\,R_5 \\ 2a &= -\,0{\cdot}742\,1\,R_1 + 7{\cdot}149\,5\,R_3 - 14{\cdot}191\,R_5 \\ b &= +\,0{\cdot}335\,5\,R_1 - 3{\cdot}511\,9\,R_3 - 3{\cdot}136\,R_5 \end{aligned}$$

the general solution of (372) is

$$\begin{aligned} E_2 &= -\frac{a}{b} \pm \left\{\frac{m}{b} + \frac{a^2}{b^2}\right\}^{\frac{1}{2}} \\ c &= 1 - F = \frac{0{\cdot}192\,2 - 0{\cdot}043\,82\,E_2}{R_5} \\ \delta\mu &= 0{\cdot}012\,66 - \frac{R_1\,(0{\cdot}192\,2 - 0{\cdot}043\,82\,E_2)}{R_5\,(36{\cdot}2 + 15{\cdot}403\,E_2)} \\ &\text{Mass of Moon} = 0{\cdot}012\,50 + \delta\mu \end{aligned} \qquad (373)$$

or more conveniently, and with no real loss of accuracy,

$$\begin{aligned} \delta\mu &= \frac{0{\cdot}036\,52\,S_2 - 0{\cdot}126\,58\,K_2 - 0{\cdot}004\,35\,N}{0{\cdot}930\,51\,S_2 - 10{\cdot}000\,00\,K_2 - 0{\cdot}454\,92\,N} \\ E_2 &= \frac{0{\cdot}192\,20\,S_2 - N\,(0{\cdot}458\,20 - 36{\cdot}20\delta\mu)}{0{\cdot}043\,82\,S_2 + N\,(0{\cdot}194\,96 - 15{\cdot}40\delta\mu)} \\ c &= 1 - F = \frac{(0{\cdot}192\,20 - 0{\cdot}043\,82\,E_2)\,M_2}{N} \\ &\text{Mass of Moon} = 0{\cdot}012\,50 + \delta\mu \end{aligned} \qquad (374)$$

The last solution shows that Professor FERREL's process results in replacing $1/M_2$ by an indeterminate, c/M_2; and thus, instead of having four fundamental quantities

*This c is identical with the $1 - F$ of equations (130), (345), and (354) of his Tidal Researches. (336, pp. 75, 188, and 195.)

from which to find two unknowns, he has but three fundamental quantities from which to find three unknowns.

Table XV contains a series of corrections to the Moon's mass, deduced from tidal observations made in thirty-four different ports; namely, at Brest and Boston of FERREL's list, and at all the stations named in BAIRD and DARWIN's list of harmonic

TABLE XV.—*Corrections to the Moon's Mass, deduced from Observations of the Tides.*

Reference Number.	Name of Station.	Duration of Observations.	$\delta\mu'$	$\delta\mu''$	$\delta\mu'''$	$\frac{1}{2}(\delta\mu' + \delta\mu'')$
		Years.				
1	Penobscot Bay, Me	6	− 0·000 50	+ 0·004 39	+ 0·002 67	+ 0·001 94
2	Boston, Mass. (336, p. 196.)	19	. . . . .	. . . . .	. . . . .	− 0 26
3	Sandy Hook, N. J	6	+ 0 59	+ 3 54	+ 2 35	+ 2 07
4	St. Thomas, West Indies	3	+ 3 21	. . . . .	. . . . .	. . . . .
5	Falkland Islands, South Atlantic . . .	1	− 2 65	+ 2 78	+ 3 59	+ 0 06
6	San Diego, Cal	5	− 0·000 71	− 0·001 36	+ 0·001 24	− 0·001 03
7	Fort Point, San Francisco, Cal	3	+ 0 94	+ 3 82	+ 3 86	+ 2 38
8	Astoria, Oregon	3	+ 2 23	+ 2 96	+ 1 14	+ 2 60
9	Port Townsend, Wash	3	+ 0 49	+ 2 95	+ 1 69	+ 1 72
10	Freemantle, West Australia	1	+ 3 87	− 5 06	+ 4 62	− 0 60
11	Port Blair, Andaman Islands	3	− 0·001 31	+ 0·000 07	+ 0·000 99	− 0·000 62
12	Moulmein, Burmah	3	+ 0 84	+ 0 95	− 1 77	+ 0 90
13	Amherst, Burmah	3	+ 3 33	+ 2 42	+ 4 89	+ 2 87
14	Rangoon, Burmah	3	+ 4 74	+ 2 18	+ 0 70	. . . . .
15	Kidderpore, near Calcutta	2	+ 2 95	+ 2 12	+ 1 29	. . . . .
16	Diamond Harbor, river Hooghly . . .	2	− 0·002 09	+ 0·001 26	+ 0·001 01	− 0·000 42
17	Dublat, Saugor Is., mouth of Hooghly .	2	+ 0 02	− 0 34	+ 0 58	− 0 16
18	False Point, Bengal	2	− 2 35	− 2 00	− 1 56	− 2 17
19	Vizagapatam	4	+ 2 56	+ 0 34	+ 1 99	+ 1 45
20	Madras	3	− 1 07	− 2 36	− 1 08	− 1 72
21	Negapatam	2	− 0·003 61	− 0·000 05	+ 0·000 73	− 0·001 83
22	Paumben	4	− 3 70	+ 1 11	+ 1 31	− 1 29
23	Beypore	5	+ 2 50	− 0 94	− 1 83	+ 0 78
24	Karwar	5	+ 2 27	− 0 59	+ 0 76	+ 0 84
25	Bombay	5	+ 1 58	− 3 34	− 0 98	− 0 88
26	Karachi	15	+ 0·001 21	− 0·000 52	+ 0·001 46	+ 0·000 35
27	Aden	4	+ 1 10	− 3 41	+ 1 19	− 1 16
28	Port Louis, Mauritius	1	+ 4 53	− 4 90	+ 5 27	− 0 18
29	Brest, France. (336, p. 189.)	19	. . . . .	. . . . .	. . . . .	+ 0 32
30	Helbre Island, mouth of river Dee . . .	10	− 2 20	+ 2 25	+ 0 95	+ 0 02
31	Liverpool, England	4	− 0·001 08	+ 0·002 77	+ 0·001 80	+ 0·000 84
32	Liverpool, England	3	− 1 38	+ 2 40	+ 1 23	+ 0 51
33	Portland Breakwater, England	4	− 2 72	− 2 78	+ 0 43	− 2 75
34	West Hartlepool, England	3	+ 2 19	+ 1 88	+ 0 70	+ 2 04

tidal constants where the observations extended over a period of two or more years, together with some where they were limited to a single year. The corrections given in the columns headed $\delta\mu'$, $\delta\mu''$, $\delta\mu'''$ have been computed, respectively, by means of formulæ (367), (371), and (374), from BAIRD and DARWIN's harmonic constants:* and as the two last-mentioned formulæ deal with the same data, we have now to consider which of them should be preferred.

At St. Thomas the K_2 and N tides seem unknown, or at least they are not given in BAIRD and DARWIN's list. Rangoon is situated upon the eastern arm of the Irrawaddy River, at a distance of 21 miles from the sea; and Kidderpore is a suburb of Calcutta, on the Hooghly River, 80 miles from the sea, the width of the navigable channel for 10 miles below that point being only about 250 yards. These circumstances render it probable that the values of $\delta\mu$ found from the tides at St. Thomas, Rangoon and Kidderpore are untrustworthy, and accordingly we shall reject them. An examination of the remaining data reveals facts which may be tabulated thus:

Column	$\delta\mu'$	$\delta\mu''$	$\delta\mu'''$
Number of plus corrections	16	16	24
Number of minus corrections . .	13	13	5
Sum of corrections	+4·88	+8·24	+39·23

If we divide each of these sums by 29, and add the quotients respectively to 0·012 50, the resulting values of the Moon's mass will be 1 : 78·9, 1 : 78·2 and 1 : 72·2. The latter is so much too large as to place its erroneous character beyond doubt, and we therefore conclude that formula (371) is decidedly preferable to (374). The failure of (374) probably indicates that the height of the tides is influenced not so much by the direct effect of friction upon their amplitudes as by its indirect effect arising from the changes which it produces in the epochs of certain shallow-water components, which cannot be separated from the deep-water components with which they are combined;† and that hypothesis is further supported by the known fact that when the effects of friction and of the Earth's rotation can be regarded as of the second order, their effect upon the epochs of the oscillations is also of the second order, but upon the amplitudes it is of the third order only.‡ The great difference between the sums of the corrections in the columns $\delta\mu'$ and $\delta\mu''$ is probably due to the circumstance that the diurnal tides are affected by fewer shallow-water components than the semi-diurnal ones.

The wide range in the values of $\delta\mu'$ and $\delta\mu''$ shows the imperfection of our present tidal theory, and Professor FERREL thinks its improvement depends mainly upon the study of the shallow-water terms. He adds:§

> With regard to the determination of the Moon's mass, from the results so far as obtained the relations of the diurnal tides promise better success in the future than those of the semi-diurnal tides. The diurnal tides are not affected by so many shallow-water components, and it is probable that these can be determined from the analysis of the observations, since there are two comparatively quite large components with periods differing from those of any others, and hence can be determined by analysis of the observations; and then from the theoretical relations

* 327, pp. 188–207. † 336, p. 55. ‡ 336, p. 51. § 338, p. 299.

given in Schedule III the others can be, at least approximately, determined, and the components of deep-water tides which they affect can be corrected for their effect. The relations of these corrected results, obtained from the analysis of the observations, should then agree with the theoretical relatives, and give a correct mass of the Moon.

Instead of attempting such intricate and uncertain computations, we shall employ another method which is simpler and not less promising. It is evident from Table XV that the value of the Moon's mass given by the tides varies irregularly, not only from point to point along the coast, but also within rivers and large harbors, and even with respect to the diurnal as distinguished from the semi-diurnal tidal components; $\delta\mu'$ and $\delta\mu''$ having the same sign at 15 places and opposite signs at 16 places. It is therefore highly probable that the cause of these variations may justly be regarded as an accidental one, whose effect will vanish from the mean of a series of observations made at a sufficient number of stations; and if so, the best mode of treating the corrections in Table XV will be to take the mean of those given in the fourth and fifth columns. By that method the result for each station will be $\frac{1}{2}(\delta\mu' + \delta\mu'')$, as given in the last column of the table; and then the question of weights arises. At first sight observations extending over a long period seem likely to be more exact than those limited to a single year, but in reality there is little difference between them, because the accidental errors at any station are generally small compared with those due to constant causes. The arithmetical mean will therefore be the most probable result, and after rejecting St. Thomas, Rangoon and Kidderpore, we have from the remaining 29 stations in the fourth and fifth columns

$$\begin{aligned} \delta\mu' &= +0{\cdot}000\,168 \pm 0{\cdot}000\,305 \\ \delta\mu'' &= +0{\cdot}000\,284 \pm 0{\cdot}000\,348 \end{aligned} \qquad (375)$$

Regarded as a single series, the 58 corrections in the fourth and fifth columns give

$$\tfrac{1}{2}(\delta\mu' + \delta\mu'') = +0{\cdot}000\,226 \pm 0{\cdot}000\,230 \qquad (376)$$

while the 31 corrections in the last column give

$$\tfrac{1}{2}(\delta\mu' + \delta\mu'') = +0{\cdot}000\,214 \pm 0{\cdot}000\,183 \qquad (377)$$

As (377) contains corrections from Boston and Brest which are not included in (376), these two results are not strictly comparable, but doubtless the probable error of the former has been somewhat diminished by the circumstance that it is based upon the value of $\frac{1}{2}(\delta\mu' + \delta\mu'')$ for each station, instead of upon the entire series of values of $\delta\mu'$ and $\delta\mu''$. To estimate what probable error (377) would have had in the latter case, we may imagine the number of corrections in (376) increased from 58 to 62 while their individual accuracy remains unchanged, and thus we shall find

$$\pm 0{\cdot}000\,230 \surd(58/62) = \pm 0{\cdot}000\,222 \qquad (378)$$

In order to ascertain whether any of the corrections are abnormally large Peirce's criterion was applied, and the limit for the rejection of one observation was found to be 0·006 79 in forming (376), and 0·003 70 in forming (377). All the corrections are therefore well within the limit, and none should be rejected. Nevertheless, it may be thought that with the comparatively small number of observations available, the final result is injuriously affected by the larger corrections, and to test that theory all corrections greater than 0·002 00 were arbitrarily rejected from the last column of the table, after which (377) became

$$\tfrac{1}{2}(\delta\mu' + \delta\mu'') = -0{\cdot}000\,018 \pm 0{\cdot}000\,146 \qquad (379)$$

Very likely (379) is nearer the truth than (377), but as no satisfactory reason can be assigned for rejecting any of the corrections, we shall adopt for $\delta\mu$ the value (377) with the probable error (378) We therefore have as our final result from the tides

$$\delta\mu = +0{\cdot}000\,214 \pm 0{\cdot}000\,222$$
$$\text{Mass of Moon} = 0{\cdot}012\,714 \pm 0{\cdot}000\,222 = \frac{1}{78{\cdot}653 \pm 1{\cdot}374} \qquad (380)$$

The mass of the Moon employed in previous sections was derived from the lunar parallax, and if the probable error of the observed value of that quantity is taken to be $\pm 0{\cdot}202\,74''$, as given in Table VI, the resulting probable error of the mass is $\pm 0{\cdot}000\,180$: whence it follows that the mass from the tides has only one-third less weight than that from the parallax.

27.—A MORE COMPREHENSIVE LEAST SQUARE ADJUSTMENT.

The discussion in Sections 25 and 26 shows that the flattening of the Earth, and the mass of the Moon from the tides, should be included in the least square adjustment, and we have now to develop the equations necessary for that purpose.

For the adjustment of the flattening a symbolic correction to its observed value must be introduced in the conditional equations, and we shall do that by the indirect method employed on page 111, because any other process would entail needless complications. The requisite numerical data corresponding to the two values of ε used on page 111 are given in Table XVI; those for $\varepsilon' = 1/293{\cdot}5$ having been collected from pages

TABLE XVI.—*Quantities employed in forming the Conditional Equations* (382).

ε	1 : 293·466 3	1 : 300·000 0
e^2	0·006 803 481 0	0·006 655 555 6
a	20 926 202 feet.	20 925 320 feet.
$l_{35 \cdot 3}$	3·256 872 feet.	3·256 831 feet.
ρ^2	0·997 742 482	0·997 791 32
$\sigma\sqrt{\frac{2}{3}}$	0·002 310 461 5	0·002 310 336 1
Log. coefficient of 1st equation	2·784 9932·2	2·784 9818·8
2d equation	5·303 1248 — 10	5·303 1248 — 10
3d equation	4·681 9624 — 10	4·681 9624 — 10
4th equation	8·912 4816·0	8·912 4633·1
5th equation	7·526 0362·0	7·526 0178·9
Coefficients of 6th equation	216 236·648	216 253·60
	3·757 444 92	3·757 446 6
	807 952·643	808 015·98
Log. coefficient of 7th equation	4·665 0707·1 — 10	4·665 0366·5 — 10

57, 61, 63, 105, and 106, and those for $\varepsilon'' = 1/300$ having been specially computed. The coefficients in the second column of the table agree with those of the equations (185), both having been computed with the value ε' of the flattening; but the coefficients in the third column do not agree with those of the equations (338), although both were computed with the value ε'' of the flattening. The difference arises from the circumstance that in (338) the equatorial semi-diameter of the Earth is regarded as independent of the flattening, while in Table XVI that assumption is abandoned and a has been found from (356), with $d\varepsilon = 1/300 - 1/293{\cdot}466\,3 = -0{\cdot}000\,074\,213$. Then e^2, ρ^2, and $l_{35\cdot3}$ were found from (326), (7), and (323), and the remaining quantities from (152), (153), (162), (181), and (183).

For the direct adjustment of the Moon's mass it is only necessary to modify the numerical coefficient of (157), and to include the corrected expression among the conditional equations. It is the last of the group (382).

If now, in accordance with (361), we put

$$d\varepsilon = \varepsilon - 1/293{\cdot}4663 = \varepsilon - 0{\cdot}003\,407\,546 \qquad (381)$$

and let x represent any one of the coefficients in Table XVI, then the numerical values given for that coefficient in the second and third columns of the table will be respectively x' and x'', and their substitution in (363) will give the general expression for x in terms of $d\varepsilon$. In that way the following set of conditional equations for the more comprehensive least square adjustment was obtained:

$$0 \text{ or } v_1 = p - [2\,784\,993\,22 + 0{\cdot}152\,8d\varepsilon]\left(\frac{E}{1+M}\right)^{\frac{1}{3}}$$

$$0 \text{ or } v_2 = p - [5{\cdot}303\,124\,8 - 10]\,PQ\,\frac{1+M}{1-M}$$

$$0 \text{ or } v_3 = p - [4{\cdot}681\,962\,4 - 10]\,PL\,\frac{1+M}{M}$$

$$0 \text{ or } v_4 = p - \frac{[8\,912\,481\,60 + 0\,246\,4d\varepsilon]}{V\theta}$$

$$0 \text{ or } v_5 = p - \frac{[7{\cdot}526\,036\,20 + 0{\cdot}246\,7d\varepsilon]}{V\alpha}$$

$$0 \text{ or } v_6 = \mathfrak{A} - \mathfrak{L}\left\{\frac{1 - (216\,236{\cdot}65 - 228\,400d\varepsilon)\sin^3 P}{3{\cdot}757\,444\,9 - 0{\cdot}022\,91d\varepsilon - (807\,952{\cdot}64 - 853\,360d\varepsilon)\sin^3 P}\right\}$$

$$0 \text{ or } v_7 = 1 + M - \frac{[4{\cdot}665\,070\,71 + 0{\cdot}458\,9d\varepsilon - 10]}{\sin^3 P}$$

Proceeding as in Section 21, we have next to differentiate these equations with respect to all the quantities expressed symbolically, and in doing so it must be remembered that (204) and (70) require that when E becomes $E + dE$, $\mathfrak{L}$ shall become $\mathfrak{L} - 31\,716''dE$, and when ε becomes $\varepsilon + d\varepsilon$, P shall become $P + 5\,062''d\varepsilon$. The results are

$$
\left.
\begin{aligned}
0 &= v_1 + dp + \frac{p}{3(1+M)}dM - \frac{p}{3E}dE - 0{\cdot}352\,5\,p\,.\,d\varepsilon \\
0 &= v_2 + dp - \frac{2p}{1-M^2}dM - \frac{p}{P}(dP + 5\,062''\,d\varepsilon) - \frac{p}{Q}dQ \\
0 &= v_3 + dp + \frac{p}{M+M^2}dM - \frac{p}{P}(dP + 5\,062''\,d\varepsilon) - \frac{p}{L}dL \\
0 &= v_4 + dp + \frac{p}{V}dV + \frac{p}{\theta}d\theta - 0{\cdot}567\,8\,p\,.\,d\varepsilon \\
0 &= v_5 + dp + \frac{p}{V}dV + \frac{p}{\alpha}d\alpha - 0{\cdot}567\,8\,p\,.\,d\varepsilon \\
0 &= v_6 + d\mathfrak{A} - \frac{\mathfrak{A}}{\mathfrak{L}}(d\mathfrak{L} - 31\,716''\,dE) + [2{\cdot}913\,3]\,d\varepsilon \\
&\quad + 3\,\text{arc}\,1''\,(\mathfrak{L} - [0{\cdot}572\,46]\mathfrak{A})\frac{\sin^2 P\cos P\,.\,dP}{[5{\cdot}239\,96-10]-[0{\cdot}572\,46]\sin^3 P} \\
0 &= v_7 + dM + 3\,\text{arc}\,1''\cot P\,(1+M)\,dP \\
&\quad + (3\,\text{arc}\,1''\cot P\,5\,062'' - 1{\cdot}056\,8)(1+M)\,d\varepsilon
\end{aligned}
\right\}\qquad (383)
$$

To secure the utmost accuracy in the subsequent computations, the coefficients in (383) should be reduced to numbers with values of the quantities involved midway between the observed and adjusted ones. The following were used, and they fulfill the required condition quite approximately:

$p = 8{\cdot}809''$	$\mathfrak{A} = 9{\cdot}224'$	$\alpha = 20{\cdot}460''$	$V = 186\,342$
$P = 3\,422{\cdot}67$	$Q = 125{\cdot}20$	$\theta = 497{\cdot}5^s$	$M = 0{\cdot}012\,520$
$\mathfrak{L} = 50{\cdot}358$	$L = 6{\cdot}518$		$E = 0{\cdot}000\,003\,030\,750$

Their substitution in (383), and the addition of the appropriate v to each equation, gave rise to the following expressions:

$$
\left.
\begin{aligned}
0 &= v_1 + dp + [0{\cdot}4624]dM - [5{\cdot}9863]dE - [0\,4921]d\varepsilon \\
0 &= v_2 + dp - [1{\cdot}2459]dM - [7{\cdot}4106-10]dP - [8{\cdot}8473-10]dQ - [1{\cdot}1149]d\varepsilon \\
0 &= v_3 + dp + [2{\cdot}8419]dM - [7{\cdot}4106-10]dP - [0{\cdot}1308]dL - [1{\cdot}1149]d\varepsilon \\
0 &= v_4 + dp + [5{\cdot}6746-10]dV + [8{\cdot}2481-10]d\theta - [0{\cdot}6991]d\varepsilon \\
0 &= v_5 + dp + [5\,6746-10]dV + [9{\cdot}6340-10]d\alpha - [0{\cdot}6991]d\varepsilon \\
0 &= v_6 + d\mathfrak{A} - [9{\cdot}2628-10]d\mathfrak{L} + [9{\cdot}3169-10]dP + [3{\cdot}7641]dE + [2\,9133]d\varepsilon \\
0 &= v_7 + dM + [6{\cdot}9481-10]dP + [0{\cdot}5343]d\varepsilon
\end{aligned}
\right\}\qquad (384)
$$

The equations (384) were employed to eliminate seven of the twelve unknowns which they contain, and in (385) each of the twelve is expressed in terms of the remaining five, namely, in terms of dp, dP, $d\mathfrak{L}$, $d\alpha$, $d\varepsilon$, and known quantities. Adopting the system of weights given by (188) for the probable errors in (392), and imagining the symbols in the first column of (385) to be replaced by zeros, the weighted normal equations (386) were formed in the usual way; but in them the differentials are inclosed in parentheses to indicate that they have now become corrections to the observed quantities. The solution of the normals gave the values of (dp), (dP), $(d\mathfrak{L})$, $(d\alpha)$, and $(d\varepsilon)$ which are contained in (387), and by substituting these values in (385) all the other corrections in (387) were obtained.

Solution of the Equations (384) in Terms of dp, dP, d𝔓, da, and dε.

[NOTE.—Instead of the numbers, their logarithms are tabulated.]

(385)

	dp	dP	$d\mathfrak{P}$	da	$d\epsilon$	Absolute terms.	Log. weight.
dp =	+ 0·000 0	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .	0·131 0
dP =	. . . . .	+ 0·000 0	. . . . .	. . . . .	. . . . .	. . . . .	7·834 4 − 10
$d\mathfrak{P}$ =	. . . . .	. . . . .	+ 0·000 0	. . . . .	. . . . .	. . . . .	1·211 0
$d\mathfrak{N}$ =	− 7·777 8 − 10	− 9·316 9 − 10	+ 9·262 8 − 10	. . . . .	− 2·913 3	$-[7{\cdot}777\,8-10]\,v_1 - v_6 + [8{\cdot}240\,2-10]\,v_7$	9·901 6 − 10
dQ =	+ 1·152 7	+ 9·268 6 − 10	. . . . .	. . . . .	+ 2·827 1	$+[1{\cdot}152\,7]\,v_2 + [2{\cdot}398\,6]\,v_7$	6·911 8 − 10
dL =	+ 9·869 2 − 10	− 9·661 0 − 10	. . . . .	. . . . .	− 3·247 8	$+[9{\cdot}869\,2-10]\,v_3 - [2{\cdot}711\,1]\,v_7$	9·591 8 − 10
da =	. . . . .	. . . . .	. . . . .	+ 0·000 0	. . . . .	. . . . .	9·917 2 − 10
$d\theta$ =	. . . . .	. . . . .	. . . . .	+ 1·385 9	. . . . .	$-[1{\cdot}751\,9]\,(v_4 - v_5)$	5·982 8 − 10
dV =	− 4·325 4	. . . . .	. . . . .	− 3·959 4	+ 5·024 5	$-[4{\cdot}325\,4]\,v_6$	2·887 4 − 10
dE =	+ 4·013 7 − 10	− 1·424 2 − 10	. . . . .	. . . . .	− 5·128 6 − 10	$+[4{\cdot}013\,7-10]\,v_1 - [4{\cdot}476\,1-10]\,v_7$	11·588 8
dM =	. . . . .	− 6·948 1 − 10	. . . . .	. . . . .	− 0·534 3	$-v_7$	3·307 3
$d\epsilon$ =	. . . . .	. . . . .	. . . . .	. . . . .	+ 0·000 0	. . . . .	4·963 9

Weighted Normal Equations.

(386)

	v_1	v_2	v_3 and v_4	v	v_6	v_7	100 ($d\epsilon$)	(dp)	(dP)/10	($d\mathfrak{P}$)	(da)
0 =	− 0·014 692	+ 0·077 911	− 5·114 5 v_3	− 1·727 0	+ 6·529 8	+ 3 624·7	+ 187·482 9	− 6·778 28	+ 45·840 5	− 1·195 9	− 0·743 54
0 =	+ 0·413 27	+ 0·164 89	+ 0·213 89 v_3	+ 34·530	+ 0·004 779 7	− 146·92	− 6·778 28	+ 36·675 1	− 1·303 53	− 0·000 875	+ 14·866
0 =	− 0·000 719	+ 0·021 533	− 1·324 4 v_3	. . . . .	+ 1·653 9	+ 938·63	+ 45·840 5	− 1·303 53	+ 12·476 36	− 0·302 90	. . . . .
0 =	− 0·000 875 39	. . . . .	. . . . .	. . . . .	− 0·146 02	+ 0·002 538 6	− 1·195 9	− 0·000 875	− 0·302 90	+ 16·282 2	. . . . .
0 =	. . . . .	. . . . .	− 0·132 01 v_4	+ 14·998	. . . . .	. . . . .	− 0·743 54	+ 14·866	. . . . .	. . . . .	+ 7·283 55

General Expressions for the Corrections by Adjustment.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$(dp) =$	− 0·066 756 6	− 0·027 018 46	− 0·011 200 90	− 0·043 508 4	− 0·632 757	− 0·030 494 0	+ 7·183 75
$(dP) =$	− 0·031 603 7	− 0·023 180 0	+ 0·579 351	− 0·031 860 0	− 0·463 239 1	− 0·452 123	− 410·801
$(d\mathfrak{P}) =$	− 0·000 083 75	− 0·000 089 22	+ 0·002 018 65	− 0·000 088 83	− 0·001 291 94	+ 0·006 318 55	− 1·432 402
$(d\mathfrak{N}) =$	+ 0·009 324 31	+ 0·009 931 75	− 0·224 728	+ 0·009 889 51	+ 0·143 828	− 0·703 395	+ 159·464
$(dQ) =$	− 0·961 547	+ 13·820 60	+ 0·034 419 3	− 0·626 788	− 9·115 581	− 0·682 535	+ 215·117
$(dL) =$	− 0·016 817 47	+ 0·001 380 30	+ 0·239 446	− 0·011 039 48	0·160 547	+ 0·619 810	− 159·626
$(da) =$	+ 0·136 148 2	+ 0·055 083 65	+ 0·024 170 01	+ 0·106 889 0	− 0·768 228	+ 0·059 728 2	− 15·591 3
$(d\theta) =$	+ 3·310 580	+ 1·339 414	+ 0·587 718	− 53·881 9	+ 37·800 76	+ 1·452 352	− 379·119
$(dV) =$	+ 173·280 7	+ 69·231 14	+ 30·377 5	− 53·505 5	− 777·725	+ 75·068 82	− 19 595·60
$(dE) =$	$+ 963\,426 \times 10^{-12}$	$- 27\,740 \times 10^{-12}$	$- 14\,823 \times 10^{-12}$	$- 44\,768 \times 10^{-12}$	$- 651\,082 \times 10^{-12}$	$- 26\,962 \times 10^{-12}$	$+ 6\,735\,644 \times 10^{-12}$
$(dM) =$	+ 0·000 063 053	+ 0·000 041 367	− 0·000 952 756	+ 0·000 040 956	+ 0·000 595 648	+ 0·001 243 04	− 0·324 036
$(d\varepsilon) =$	− 0·000 010 230	− 0·000 006 078	+ 0·000 128 182	− 0·000 003 707	− 0·000 053 938	− 0·000 245 994	− 0·091 004 4

(387)

As a preliminary step toward finding the probable errors of the adjusted quantities, we substitute in (387) the values of v_1, v_2, v_3, etc., from (384), and thus obtain the equations (388), which are to be read in the same way as those on page 67. Then, by the process described on page 71, and with the probable errors of the observed quantities from (392), the results given in Table XVII were deduced.

Reverting now to the flattening of the Earth, there can be little doubt as to what the results are from the geodetic arcs, precession and nutation, and perturbations of the Moon; and for the pendulum experiments HELMERT's reduction seems best, partly because it is based upon more data than any other, and partly in consequence of the discussion on page 110. Accordingly, the following observed values of ε, from pages 5, 97, 95, and 102, are the only ones to be considered:

Method of determination.	ε
Geodetic arcs, CLARKE	1 : 293·47 = 0·003407503
Pendulum experiments, HELMERT . .	1 : 299·26 = 0·003341576
Precession and nutation	1 : 297·67 = 0·003359425
Perturbations of the Moon	1 : 294·93 = 0·003390635

Equations No (388).

Factors.	Logarithmic coefficients for computing—					
	(dp)	(dP)	$(d𝔓)$	$(d𝔑)$	(dQ)	(dL)
(dp)	+ 9·8928 − 10	− 8·4694 − 10	− 6·6674 − 10	+ 8·7139 − 10	− 0·4985	− 8·7195 − 10
(dP)	− 6·1681 − 10	+ 9·6625 − 10	− 5·5403 − 10	+ 7·5863 − 10	− 8·1348 − 10	+ 8·1365 − 10
$(d𝔓)$	− 7·7470 − 10	− 8·9180 − 10	+ 7·0634 − 10	− 9·1100 − 10	− 9·0969 − 10	+ 9·0550 − 10
$(d𝔑)$	+ 8·4842 − 10	+ 9·6552 − 10	− 7·8006 − 10	+ 9·8472 − 10	+ 9·8341 − 10	− 9·7923 − 10
(dQ)	− 7·2790 − 10	− 7·2124 − 10	− 4·7978 − 10	+ 6·8443 − 10	+ 9·9878 − 10	+ 5·9873 − 10
(dL)	− 8·1800 − 10	+ 9·8937 − 10	+ 7·4358 − 10	− 9·4824 − 10	+ 8·6676 − 10	+ 9·5100 − 10
(da)	+ 9·4352 − 10	+ 9·2998 − 10	+ 6·7452 − 10	− 8·7918 − 10	+ 0·5938	+ 8·8396 − 10
$(d\theta)$	+ 6·8867 − 10	+ 6·7513 − 10	+ 4·1967 − 10	− 6·2433 − 10	+ 8·0452 − 10	+ 6·2910 − 10
(dV)	+ 5·5047 − 10	+ 5·3693 − 10	+ 2·8147 − 10	− 4·8613 − 10	+ 6·6633 − 10	+ 4·9091 − 10
(dE)	− 4·8096	− 4·4471	− 2·0713	+ 4·1180	− 5·9674	− 4·2988
(dM)	+ 9·5011 − 10	+ 0·8982	+ 8·4529 − 10	− 0·4996	+ 0·8584	− 0·8251
$(d\varepsilon)$	− 0·5678	+ 3·2506	− 9·4068 − 10	+ 1·4531	− 1·6844	+ 1·6113

Factors.	Logarithmic coefficients for computing—					
	(da)	$(d\theta)$	(dV)	(dE)	(dM)	$(d\varepsilon)$
(dp)	+ 9·6493 − 10	+ 1·0352	+ 2·7469	− 3·3522 − 10	+ 6·3258 − 10	− 5·7342 − 10
(dP)	+ 7·2172 − 10	+ 8·6031 − 10	+ 0·3165	− 0·6933 − 10	+ 5·4367 − 10	+ 6·1209 − 10
$(d𝔓)$	+ 8·0389 − 10	+ 9·4248 − 10	+ 1·1382	− 1·6935 − 10	+ 6·3573 − 10	− 5·6537 − 10
$(d𝔑)$	− 8·7762 − 10	− 0·1621	− 1·8755	+ 2·4308 − 10	− 7·0945 − 10	+ 6·3909 − 10
(dQ)	+ 7·5883 − 10	+ 8·9742 − 10	+ 0·6876	− 1·2904 − 10	+ 4·4640 − 10	− 3·6311 − 10
(dL)	+ 8·5141 − 10	+ 9·9000 − 10	+ 1·6133	− 2·3017 − 10	− 7·1098 − 10	+ 6·2386 − 10
(da)	+ 9·5195 − 10	− 1·2115	+ 2·5248	+ 3·4476 − 10	− 6·4090 − 10	+ 5·3659 − 10
$(d\theta)$	− 7·2770 − 10	+ 9·9795 − 10	+ 9·9765 − 10	+ 0·8991 − 10	− 3·8604 − 10	+ 2·8171 − 10
(dV)	+ 5·4950 − 10	+ 6·8809 − 10	+ 8·5943 − 10	+ 9·5171 − 20	− 2·4785 − 10	+ 1·4354 − 10
(dE)	+ 5·1192	+ 6·5051	+ 8·2239	+ 9·9702 − 10	+ 1·7314	− 0·9286
(dM)	− 9·7981 − 10	− 1·1840	− 2·9008	+ 3·4501 − 10	+ 9·9942 − 10	+ 7·2689 − 10
$(d\varepsilon)$	+ 0·4125	+ 1·7984	+ 3·5125	− 4·3034 − 10	+ 8·9156 − 10	+ 9·7111 − 10

TABLE XVII.—*Computation of the Constants required for finding the Probable Errors of the Adjusted Quantities.*

Quantity.		p	P	𝔓	𝔑	Q	L
Squares of the several terms in (209).	p	35 383	1	2	1 981	73 451	203
	P	3	42 737	2	2 179	272	2 745
	𝔓	2	0	613 76	1 021	10	79
	𝔑	1 166	256	501	110 36	5 843	48 195
	Q	4 428	3	0	598	9 354	1
	L	586	1 569	190	236 05	55	117 112
	a	89 784	48	4	4 639	186 38	578
	θ	6 175	3	0	319	12 812	40
	V	13 244	7	1	684	27 492	85
	E	10 725	2	0	444	22 182	102
	M	50	31	4	4 918	257	2 203
	ε	148	34 451	7	8 754	254	1 814
Sum		161 692	79 108	614 471	371 947	338 362	173 157
Log. sum		5·208 69	7·898 22	4·788 50	5·570 48	7·529 38	6·238 44
Log. R_a		7·604 34	8·949 11	7·394 25	7·785 24	8·764 69	8·119 22
R_a		± 0·004 021″	± 0·088 943″	± 0·002 479″	± 0·006 099″	± 0·058 169″	± 0·013 159″

Quantity.		a	θ	V	E	M	ε
Squares of the several terms in (209).	p	147 10	86 936	23·06	37 446	332	22
	P	398	235	0·06	36	1 094	25 551
	𝔓	7	4	0·00	1	32	1
	𝔑	4 475	2 646	0·71	912	19 382	759
	Q	18 399	10 879	2·91	4 667	104	2
	L	2 731	1 614	0·43	1 027	42 442	768
	a	529 67	320 48	13·56	95 061	796	7
	θ	37 256	22 029	0·93	6 537	55	0
	V	12 665	7 489	1 196·2	14 021	117	1
	E	44 627	26 388	7·23	11 345	75	2
	M	194	115	0·03	39	882	17
	ε	73	43	0·01	44	736	25 645
Sum		797 595	478 858	1 245·13	171 136	66 047	52 775
Log. sum		5·901 78	8·680 20	3·095 20	3·233 34	0·819 85	0·722 43
Log. R_a		7·950 89	9·340 10	1·547 60	1·616 67	5·409 93	5·361 22
R_a		± 0·008 931″	± 0·218 82ˢ	± 35·286	± 4 136 9	± 25 700	± 22 973

For use in the least square adjustment we adopt the arithmetical mean of these results, with an assumed probable error equal to half the difference between the greatest and least, namely:

$$\varepsilon = 0{\cdot}003\,374\,785 \pm 0{\cdot}000\,032\,964 \qquad (389)$$
$$= 1 : 296{\cdot}315\,2 \pm 2{\cdot}894\,6$$

The substitution of (389) in (381) gives

$$d\varepsilon = -0{\cdot}000\,032\,761 \tag{390}$$

whence, by (70) and (75)

$$P = 3\,423{\cdot}08'' + 5\,062''\,d\varepsilon = 3\,422{\cdot}914\,16'' \pm 0{\cdot}121' \tag{391}$$

The values of the observed quantities employed in the more comprehensive least square adjustment are those given in (143), (391), (89), (90), (66), (67), (91), (92), (101), (132), (380), and (389), all of which are collected in (392) for convenience of reference.

$$\begin{aligned}
p &= 8{\cdot}834'' \pm 0{\cdot}008\,6'' \\
P &= 3\,422{\cdot}914\,164'' + 5\,062''d\varepsilon \pm 0{\cdot}121'' \\
\mathfrak{L} &= 50{\cdot}358\,6'' - 31\,716''dE \pm 0{\cdot}002\,48'' \\
\mathfrak{N} &= 9{\cdot}233\,1'' \pm 0{\cdot}011\,2'' \\
Q &= 125{\cdot}46'' \pm 0{\cdot}35'' \\
L &= 6{\cdot}514'' \pm 0{\cdot}016'' \\
\alpha &= 20{\cdot}466'' \pm 0{\cdot}011'' \\
\theta &= 497{\cdot}0^{s} \pm 1{\cdot}02^{s} \\
V &= 186\,347 \pm 36 \text{ miles} \\
E &= 0{\cdot}000\,003\,005\,097 \pm 0{\cdot}000\,000\,016\,056 \\
M &= 0{\cdot}012\,714 \pm 0{\cdot}000\,222 \\
\varepsilon &= 0{\cdot}003\,374\,785 \pm 0{\cdot}000\,032\,964
\end{aligned} \tag{392}$$

The adjustment was effected as follows:

1. To allow for the change in the flattening from 1 : 293·5 in (382) to 1 : 296·3 in (392), the corrected conditional equations (393) were computed from (382) by substituting therein the numerical value of $d\varepsilon$ from (390).

2. The residuals, (394), were found by substituting the observed values, (392), in the corrected conditional equations, (393).

3. The corrections by adjustment, (395), were found by substituting the residuals (394) in the formulæ (387).

4. The correctness of all the numerical processes involved in passing from (382) to (395) was checked by the well-known relation

$$\begin{aligned}[pnn] + 100\,[an](d\varepsilon) + [bn](dp) + 0{\cdot}10\,[cn](dP) \\ + [dn](d\mathfrak{L}) + [en](d\alpha) - [pvv] = 0\end{aligned}$$

where $[pnn]$ is the sum of the weighted squares of the absolute terms in (385); $[an]$, $[bn]$, $[cn]$, $[dn]$, $[en]$ are the absolute terms in (386); and $[pvv]$ is the sum of the weighted squares of the corrections by adjustment in (395). The result of the check is given in (396).

5. The incompletely adjusted quantities, (397), were found by adding the corrections (395) to the observed quantities (392).

Corrected Conditional Equations.

$$
\begin{aligned}
0 \text{ or } v_1 &= p - [2{\cdot}784\,988\,2]\left(\frac{E}{1+M}\right)^{\frac{1}{3}} \\
0 \text{ or } v_2 &= p - [5{\cdot}303\,124\,8 - 10]PQ\,\frac{1+M}{1-M} \\
0 \text{ or } v_3 &= p - [4{\cdot}681\,962\,4 - 10]PL\,\frac{1+M}{M} \\
0 \text{ or } v_4 &= p - \frac{[8{\cdot}91\dot{2}\,473\,5]}{V\theta} \\
0 \text{ or } v_5 &= p - \frac{[7{\cdot}526\,028\,1]}{V\alpha} \\
0 \text{ or } v_6 &= \mathfrak{A} - \mathfrak{L}\left\{\frac{1 - 216\,244{\cdot}13\sin^3 P}{3{\cdot}757\,445\,7 - 807\,980{\cdot}60\sin^3 P}\right\} \\
0 \text{ or } v_7 &= 1 + M - \frac{[4{\cdot}665\,055\,68 - 10]}{\sin^3 P}
\end{aligned}
\qquad (393)
$$

First set of Residuals.

$$
\begin{aligned}
v_1 &= +0{\cdot}075\,184'' & v_5 &= +0{\cdot}030\,149'' \\
v_2 &= -0{\cdot}018\,582 & v_6 &= +0{\cdot}079\,261 \\
v_3 &= +0{\cdot}294\,925 & v_7 &= +0{\cdot}000\,662 \\
v_4 &= +0{\cdot}007\,374 & &
\end{aligned}
\qquad (394)
$$

First Approximation to the Corrections by Adjustment.

$$
\begin{aligned}
(dp) &= -0{\cdot}024\,879\,56'' & (d\alpha) &= -\ 0{\cdot}011\,619\,52' \\
(dP) &= -0{\cdot}153\,067\,62 & (d\theta) &= +\ 1{\cdot}003\,814\,53^s \\
(d\mathfrak{L}) &= +0{\cdot}000\,103\,67 & (dV) &= -\ 10{\cdot}163\,88 \text{ miles} \\
(d\mathfrak{A}) &= -0{\cdot}011\,538\,96 & (dE) &= +\ 0{\cdot}000\,000\,050\,940 \\
(dQ) &= -0{\cdot}510\,094\,79 & (dM) &= -\ 0{\cdot}000\,374\,748 \\
(dL) &= +0{\cdot}007\,861\,33 & (d\varepsilon) &= -\ 0{\cdot}000\,044\,248\,41
\end{aligned}
\qquad (395)
$$

$[pmn]$	$+0{\cdot}045\,552\,43$	
$100\,[an](d\varepsilon)$	$-\ \ 5\,990\,91$	
$[bn](dp)$	$-\ \ 25\,754\,43$	
$0{\cdot}10\,[cn](dP)$	$-\ \ 5\,532\,81$	
$[dn](d\mathfrak{L})$	$-\ \ 1\,21$	
$[en](d\alpha)$	$-\ \ 5\,244\,40$	(396)
Sum	$+0{\cdot}003\,028\,67$	
$[pvv]$	$+0{\cdot}003\,028\,10$	
Check	$0{\cdot}000\,000\,57$	

First Approximation to the Adjusted Quantities.

$$
\begin{array}{ll}
p = 8''\!\cdot\!809\,120'' & \alpha = 20\cdot454\,380'' \\
P = 3\,422\cdot537\,113 & \theta = 498\cdot004^{s} \\
\mathfrak{P} = 50\cdot357\,088 & V = 186\,336\cdot84 \text{ miles} \\
\mathfrak{N} = 9\cdot221\,561 & E = 0\cdot000\,003\,056\,037 \\
Q = 124\cdot949\,91 & M = 0\cdot012\,339\,25 \\
L = 6\cdot521\,861 & \varepsilon = 0\cdot003\,330\,537
\end{array}
\qquad (397)
$$

6. By substituting the value of ε from (397) in (381), the corresponding value of $d\varepsilon$ was found to be $-0\cdot000\,077\,009$, and by substituting that in (382) the corrected conditional equations (398) were obtained.

7. When the quantities (397) were substituted in the conditional equations (398), they gave rise to a second set of residuals (399), thus showing the adjustment to be incomplete. (396) proves that no error exists in any of the numerical operations, and it is easily seen that these residuals arise from the neglect of terms of the second order in forming the differential equations (383).

8. The better adjusted quantities (400) were obtained by substituting the residuals (399) in the formulæ (387), and adding the corrections so found to the quantities (397).

9. By substituting the value of ε from (400) in (381), the corresponding value of $d\varepsilon$ was found to be $-0\cdot000\,076\,492$, and by substituting that in (382), the corrected conditional equations (401) were obtained.

10. A third set of residuals, (402), was found by substituting the quantities (400) in the conditional equations (401).

Second set of Corrected Conditional Equations.

$$
\begin{aligned}
0 \text{ or } v_1 &= p - [2\cdot784\,981\,45]\left(\frac{E}{1+M}\right)^{\frac{1}{3}} \\
0 \text{ or } v_2 &= p - [5\cdot303\,124\,8 - 10]\,PQ\,\frac{1+M}{1-M} \\
0 \text{ or } v_3 &= p - [4\cdot681\,962\,4 - 10]\,PL\,\frac{1+M}{M} \\
0 \text{ or } v_4 &= p - \frac{[8\cdot912\,462\,62]}{V\theta} \\
0 \text{ or } v_5 &= p - \frac{[7\cdot526\,017\,20]}{V\alpha} \\
0 \text{ or } v_6 &= \mathfrak{N} - \mathfrak{P}\left\{\frac{1 - 216\,254\cdot24\sin^3 P}{3\cdot757\,446\,7 - 808\,018\cdot36\sin^3 P}\right\} \\
0 \text{ or } v_7 &= 1 + M - \frac{[4\cdot665\,035\,37 - 10]}{\sin^3 P}
\end{aligned}
\qquad (398)
$$

Second set of Residuals.

$$v_1 = +0{\cdot}000\,140'' \qquad v_5 = +0{\cdot}000\,008''$$
$$v_2 = +0{\cdot}000\,107 \qquad v_6 = +0{\cdot}000\,118$$
$$v_3 = +0{\cdot}004\,322 \qquad v_7 = +0{\cdot}000\,000\,06 \qquad (399)$$
$$v_4 = +0{\cdot}000\,030$$

Second Approximation to the Adjusted Quantities.

$$p = 8{\cdot}809\,050'' \qquad \alpha = 20{\cdot}454\,513''$$
$$P = 3\,422{\cdot}542\,144 \qquad \theta = 498{\cdot}005\,981^{s}$$
$$\mathfrak{P} = 50{\cdot}357\,096 \qquad V = 186\,337{\cdot}004 \text{ miles} \qquad (400)$$
$$\mathfrak{N} = 9{\cdot}220\,520 \qquad E = 0{\cdot}000\,003\,056\,095$$
$$Q = 124{\cdot}951\,244 \qquad M = 0{\cdot}012\,335\,279$$
$$L = 6{\cdot}522\,956 \qquad \varepsilon = 0{\cdot}003\,331\,054$$

Third set of Corrected Conditional Equations.

$$0 \text{ or } v_1 = p - [2{\cdot}784\,981\,53]\left(\frac{E}{1+M}\right)^{\frac{1}{3}}$$
$$0 \text{ or } v_2 = p - [5{\cdot}303\,124\,8 - 10]\,PQ\,\frac{1+M}{1-M}$$
$$0 \text{ or } v_3 = p - [4{\cdot}681\,962\,4 - 10]\,PL\,\frac{1+M}{M}$$
$$0 \text{ or } v_4 = p - \frac{[8{\cdot}912\,462\,75]}{V\theta} \qquad (401)$$
$$0 \text{ or } v_5 = p - \frac{[7{\cdot}526\,017\,33]}{V\alpha}$$
$$0 \text{ or } v_6 = \mathfrak{N} - \mathfrak{P}\left\{\frac{1 - 216\,254{\cdot}12 \sin^3 P}{3{\cdot}757\,446\,7 - 808\,017{\cdot}92 \sin^3 P}\right\}$$
$$0 \text{ or } v_7 = 1 + M - \frac{[4{\cdot}665\,035\,61 - 10]}{\sin^3 P}$$

Third set of Residuals.

$$v_1 = +0{\cdot}000\,001'' \qquad v_5 = 0{\cdot}000\,000''$$
$$v_2 = +0{\cdot}000\,001 \qquad v_6 = -0{\cdot}000\,019$$
$$v_3 = -0{\cdot}000\,040 \qquad v_7 = -0{\cdot}000\,000\,034 \qquad (402)$$
$$v_4 = 0{\cdot}000\,000$$

Third Approximation to the Adjusted Quantities.

$$p = 8{\cdot}809\,051'' \qquad \alpha = 20{\cdot}454\,512''$$
$$P = 3\,422{\cdot}542\,157 \qquad \theta = 498{\cdot}005\,947^{s}$$
$$\mathfrak{P} = 50{\cdot}357\,096 \qquad V = 186\,337{\cdot}002 \text{ miles} \qquad (403)$$
$$\mathfrak{N} = 9{\cdot}220\,537 \qquad E = 0{\cdot}000\,003\,056\,097$$
$$Q = 124{\cdot}951\,261 \qquad M = 0{\cdot}012\,335\,305$$
$$L = 6{\cdot}522\,940 \qquad \varepsilon = 0{\cdot}003\,331\,057$$

11. The finally adjusted quantities, (403), were obtained by substituting the residuals (402) in the formulæ (387), and adding the corrections so found to the quantities (400).

12. From (403) and (381) the value of $d\varepsilon$ was found to be $-0.000\,076\,489$, and the resulting conditional equations from (382) were sensibly the same as (401).

13. The substitution of the quantities (403) in the conditional equations (401) gave the final residuals (404), which show that the adjustment is sufficiently complete.

Final Residuals.

$$\begin{aligned} v_1 &= \,0.000\,000'' & v_5 &= +\,0.000\,000'' \\ v_2 &= +\,0.000\,000 & v_6 &= +\,0.000\,006 \\ v_3 &= +\,0.000\,001 & v_7 &= +\,0.000\,000\,003 \\ v_4 &= -\,0.000\,000 \end{aligned} \qquad (404)$$

The data and results of the computation outlined in (393) to (404) are given in Table XVIII, but it yet remains to explain how the probable errors attached to the various quantities were derived. Putting r'' for the probable error assumed for a quantity of weight unity in equation (188), m for the number of observation equations in (385), n for the number of unknowns they contain, and $[pvv]$ for the sum of the weighted squares of the corrections by adjustment, we have in accordance with the procedure described on pages 70 and 71

$$q = \frac{0.674\,49}{r''}\sqrt{\left(\frac{[pvv]}{m-n}\right)} \qquad (405)$$

With $r'' = 0.01$, $m = 12$, $n = 5$, and the values of p and v from (385) and the third column of Table XVIII, (405) gives $q = 1.409\,1$. The probable errors in Table XVIII result from the multiplication of that value of q into the respective probable errors in (392) and the respective values of R_a in Table XVII.

The following explanations relate to the quantities appended to Table XVIII:

The masses of Mercury, Venus, and the Earth, together with their probable errors, were computed by means of formulæ (344), (345), and (346), with the value of $dE \pm dE'$ given in the fourth column of Table XVIII. The mass of the Moon is that given in the table, transformed from a decimal to a vulgar fraction.

The lengths of the equatorial and polar semi-diameters of the Earth were found from (356), with the value of ε from the fourth column of Table XVIII, and $d\varepsilon$ from (381). As the expressions (356) are of the form $a = m + n.d\varepsilon$, if we put r with a subscript letter for the probable error of the quantity symbolized by the subscript, we shall have

$$r_a^2 = r_m^2 + (nr_\varepsilon)^2 \qquad (406)$$

TABLE XVIII.—*Final Results for the Epoch* 1850.0.

Quantities.	Observed values.	Corrections by adjustment.	Adjusted values.
	″ ″	″	″ ″
p	8·834 ± 0·012 12	− 0·024 95	8·809 05 ± 0·005 67
P	3 422·692 81 ± 0·170 50	− 0·150 65	3 422·542 16 ± 0·125 33
$\mathfrak{P}$	50·356 99 ± 0·003 49	+ 0·000 11	50·357 10 ± 0·003 49
$\mathfrak{N}$	9·233 1 ± 0·015 78	− 0·012 56	9·220 54 ± 0·008 59
Q	125·46 ± 0·493 18	− 0·508 74	124·951 26 ± 0·081 97
L	6·514 ± 0·022 55	+ 0·008 94	6·522 94 ± 0·018 54
a	20·466 ± 0·015 50	− 0·011 49	20·454 51 ± 0·012 58
θ	497·0^s ± 1·437 28^s	+ 1·005 95^s	498·005 95^s ± 0·308 34^s
V	186 347 ± 50·728 miles	− 9·998 miles	186 337·00 ± 49·722 miles
E	0·000 003 005 097 ± 0·000 000 022 625	+ 51 000	0·000 003 056 097 ± 0·000 000 005 829
M	0·012 714 ± 0·000 312 820	− 378 695	0·012 335 305 ± 0·000 036 214
ε	0·003 374 785 ± 0·000 046 450	− 43 728	0·003 331 057 ± 0·000 032 371

$$\text{Mass of Mercury} = \frac{0·358\ 223 \pm 0·072\ 441}{3\ 000\ 000} = \frac{1}{8\ 374\ 672 \pm 1\ 765\ 762}$$

$$\text{Mass of Venus} = \frac{0·982\ 587 \pm 0·004\ 503}{401\ 847} = \frac{1}{408\ 968 \pm 1\ 874}$$

$$\text{Mass of Earth} = \frac{1·084\ 720 \pm 0·002\ 069}{354\ 936} = \frac{1}{327\ 214 \pm 624}$$

$$\text{Mass of Moon} = \frac{1}{81·068 \pm 0·238}.$$

Earth's equatorial semidiameter = 20 925 293 ± 409·4 feet,
= 3 963·124 ± 0·078 miles.
Earth's polar semidiameter = 20 855 590 ± 325·1 feet,
= 3 949·922 ± 0·062 miles.
One Earth quadrant = 393 775 819 ± 4 927 inches,
= 32 814 652 ± 410·6 feet,
= 6 214·896 ± 0·078 miles.

$$\text{Earth's flattening} = \frac{a-b}{a} = \frac{1}{300·205 \pm 2·964}$$

Mean distance from Earth to Sun = 92 796 950 ± 59 715 miles.
Mean distance from Earth to Moon = 238 854·75 ± 9 916 miles.
Length of seconds pendulum = 3·251 045 + 0·017 356 $\sin^2 \varphi$ feet,
= 39·012 540 + 0·208 268 $\sin^2 \varphi$ inches.
Acceleration by gravity, per second of mean time
= 32·086 528 + 0·171 293 $\sin^2 \varphi$ feet,
= 9·779 886 + 0·052 210 $\sin^2 \varphi$ meters.

Table XVIII gives $r_\varepsilon = \pm 0{\cdot}000\,032\,371$, and upon the assumption that the probable error of the length of a well measured geodetic arc is about one part in 150 000, (356) gives

$$r_{\rm m} \text{ for } a = \pm\ 139{\cdot}5 \text{ feet} \qquad\qquad r_{\rm m} \text{ for } b = \pm\ 139{\cdot}0 \text{ feet}$$

With these values the probable errors attached to the adjusted lengths of the Earth's semidiameters were found from (406).

A quadrant of the Earth, measured from the north pole along any meridian to the equator, is the theoretical basis of the metric system, and in computing its length from our values of the Earth's polar and equatorial semidiameters the following formula was employed:

One Earth Quadrant

$$= \frac{\pi(a+b)}{4}\left\{1 + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^2 + \frac{1}{64}\left(\frac{a-b}{a+b}\right)^4 + \frac{1}{256}\left(\frac{a-b}{a+b}\right)^6 + \text{etc.}\right\} \qquad (407)$$

The distances of the Sun and Moon from the Earth, together with their probable errors, were computed by means of the formula

$$\mathrm{D} = a \operatorname{cosec} p \pm \sqrt{[(\operatorname{cosec} p \,.\, r_{\rm a})^2 + (\mathrm{D} \cot p \text{ arc } 1'' \, r_{\rm p})^2]} \qquad (408)$$

where a is the Earth's equatorial semidiameter, $r_{\rm a}$ its probable error, D the distance corresponding to the parallax p, and $r_{\rm p}$ the probable error of p. The expression for the probable error of D was found in the usual way, by differentiating $a \operatorname{cosec} p$, squaring the several terms, and replacing the differentials by the probable errors.

The equations (321) and (12) give

$$l = \left(l_{45} - \frac{5a}{t_1^2}\right)\frac{1 - \varepsilon \sin^2\varphi}{1 - \frac{1}{2}\varepsilon} + \frac{10a}{t_1^2}\sin^2\varphi \qquad (409)$$

from which the length of the seconds pendulum was computed with the adjusted values of a and ε from Table XVIII, l_{45} from (10) and t_1 from (17). The expression for the acceleration by gravity then followed from the usual formula, $g = \pi^2 l$.

Replacing $\sin^2\varphi$ in (409) by its equivalent, $\frac{1}{2}(1 - \cos 2\varphi)$, we get

$$l = l_{45} + \frac{1}{2}\left\{\left(l_{45} - \frac{5a}{t_1^2}\right)\frac{\varepsilon}{1 - \frac{1}{2}\varepsilon} - \frac{10a}{t_1^2}\right\}\cos 2\varphi \qquad (410)$$

the differential of which is

$$\begin{aligned} dl = &+ \left\{1 + \frac{\varepsilon}{2-\varepsilon}\cos 2\varphi\right\}dl_{45} - \frac{5da}{t_1^2}\left\{1 + \frac{\varepsilon}{2-\varepsilon}\right\}\cos 2\varphi \\ &+ \left\{l_{45} - \frac{5a}{t_1^2}\right\}\frac{2\,d\varepsilon}{(2-\varepsilon)^2}\cos 2\varphi \end{aligned} \qquad (411)$$

or, after substituting the numerical values of l_{45}, a, ε, and t_1,

$$\begin{aligned} dl = &+ (1 + 0{\cdot}001\,668\,307 \cos 2\varphi)dl_{45} \\ &- 0{\cdot}000\,000\,000\,675 \cos 2\varphi \,.\, da \\ &+ 1{\cdot}628\,234 \cos 2\varphi \,.\, d\varepsilon \text{ foot} \end{aligned} \qquad (412)$$

Then writing r_l for the probable error of l, r_0 for that of l_{45}, and substituting the numerical probable error of ε from Table XVIII, we find with sufficient accuracy

$$r_l = \pm\sqrt{(r_0^2 + 0{\cdot}000\,000\,002\,778 \cos^2 2\varphi)} \text{ foot} \qquad (413)$$

The value of r_0 seems to be about $\pm 0{\cdot}000\,040$ of a foot, but its uncertainty is so great that no probable error is appended either to the length of the seconds pendulum or to the expression for the acceleration by gravity.

Formulæ (408) and (413) are sufficient to give the probable errors of D and l with all needful exactness, but nevertheless they are only approximate. Rigorous accuracy would require the differentials of the adjusted quantities to be expressed in terms of the original observed quantities, and then the formulæ would become much more complicated.

28.—THE ELECTRIC CONSTANT v.

MAXWELL has employed the symbol v to denote the ratio of the electrostatic to the electromagnetic unit of electricity, and has endeavored to prove that v is a velocity identical with the velocity of light. The correctness of his theory is now generally admitted, but its experimental verification is still a matter of interest, and for that reason an attempt has been made to collect all the published values of v in Table XIX.

TABLE XIX.—*Values of the Electric Constant v, found by various Experimenters.*

Date.	Authority.	Observed v	Corrected v
1857	KOHLRAUSCH & WEBER (359, p. 264)* . . .	310·7	310·7
1869	W. THOMSON & KING (360 and 362, p. 417) .	284·6	280·8
1868	MAXWELL (361, p. 651)	288·0	284·1
1873	M'KICHAN (362, p. 427)	293·2	289·3
1879	AYRTON & PERRY (363, p. 141)	298·0	296·0
1879	HOCKIN (364, p. 288)	298·8	296·8
1880	SHIDA (365, p. 436)	299·5	295·5
1882	EXNER (367, p. 111, and 369, p. 301)	301·1	292·3
1883	J. J. THOMSON (368, p. 721)	296·3	296·3
1884	KLEMENČIČ (369, p. 328)	301·9	301·9
1888	HIMSTEDT (370, p. 136)	300·9	300·9
1889	W. THOMSON (372, p. 312)	. . .	300·4
1889	ROWLAND (371, p. 296)	298·2	298·2
1889	ROSA (372, p. 312)	300·0	300·0
1890	J. J. THOMSON & SEARLE (373, p. 378) . . .	299·6	299·6

*The c of KOHLRAUSCH & WEBER is $v\sqrt{2}$.

The results obtained by the several experimenters are given in the third column of the table, but they depend upon various standards of resistance, and must be reduced to absolute measure before comparison either with each other or with the velocity of light. That reduction has been effected upon the assumption that the true ohm is 0·986 64 of the British Association ohm, and the results are given in the last column of the table. Since 1857 the observed values of v have approximated more and more to the velocity of light, and if we reject the results obtained by KOHLRAUSCH & WEBER, Sir W. THOMSON & KING, MAXWELL, M'KICHAN, and EXNER, the ten remaining determinations give

$$v = (29\,856 \pm 155)\ 10^6 \text{ centimeters per second} \qquad (414)$$

while the velocity of light from Table XVIII is

$$V = (29\,988 \pm 8)\ 10^6 \text{ centimeters per second.} \qquad (415)$$

The difference of these results is only 0·440 of one per centum, and is so much less than the square root of the sum of the squares of the probable errors as to leave little doubt that V and v are absolutely identical.

29.—SUPPLEMENTARY DATA.

The following data are inserted here to remedy omissions which have occurred in the preceding pages from various causes.

In 1863 Archdeacon PRATT proposed a method of deducing the size and figure of the Earth, which was based upon the idea of eliminating the effects of local attraction from the amplitudes of the geodetic arcs employed, and by it in 1871 he found from the Anglo-Gallic, Russian, and Indian arcs*

$$a = 20\,926\,184 \text{ feet}$$
$$b = 20\,855\,304 \text{ feet}$$
$$\varepsilon = 1 / 295{\cdot}2$$

In 1876, from a series of pendulum observations made at 73 stations, whose latitudes ranged between $+79°\ 50'$ and $-62°\ 56'$, Dr. A. FISCHER found†

$$l = 0{\cdot}991\,011 + 0{\cdot}005\,105 \sin^2 \varphi \text{ meter}$$

In 1884, from the series of observations used by Dr. FISCHER, Dr. G. W. HILL found‡

$$\begin{aligned} l = \ & 0{\cdot}992\,714\,8 \text{ meter} \\ & + 0{\cdot}005\,089\,0\,\rho^{-4}\left(\sin^2\varphi - \frac{1}{3}\right) \\ & + 0{\cdot}000\,097\,9\,\rho^{-4}\cos^2\varphi\cos(2\varpi' + 29°\ 04') \\ & - 0{\cdot}000\,135\,5\,\rho^{-5}\left(\sin^3\varphi - \frac{3}{5}\sin\varphi\right) \\ & + 0{\cdot}000\,542\,1\,\rho^{-5}\left(\sin^2\varphi - \frac{1}{5}\right)\cos\varphi\cos(\varpi' + 217°\ 51') \\ & + 0{\cdot}000\,264\,0\,\rho^{-5}\sin\varphi\cos^2\varphi\cos(2\varpi' + 4°\ 49') \\ & + 0{\cdot}000\,124\,8\,\rho^{-5}\cos^3\varphi\cos(3\varpi' + 110°\ 24') \\ & + 0{\cdot}000\,148\,9\,\rho^{-6}\left(\sin^4\varphi - \frac{6}{7}\sin^2\varphi + \frac{3}{35}\right) \end{aligned} \qquad (416)$$

*15½, p. 177. Compare also 25⅞ and 22½. †24¼, col. 87. ‡57⅓, p. 339.

$$+0\cdot000\,738\,6\,\rho^{-6}\left(\sin^3\varphi - \frac{3}{7}\sin\varphi\right)\cos\varphi\cos(\varpi' + 3°\,02')$$
$$+0\cdot000\,217\,5\,\rho^{-6}\left(\sin^2\varphi - \frac{1}{7}\right)\cos^2\varphi\cos(2\varpi' + 262°\,17') \qquad (416)\ \text{Cont'd.}$$
$$+0\cdot000\,312\,6\,\rho^{-6}\sin\varphi\cos^3\varphi\cos(3\varpi' + 148°\,20')$$
$$+0\cdot000\,058\,4\,\rho^{-6}\cos^4\varphi\cos(4\varpi' + 248°\,19')$$

where φ is the geocentric latitude, ϖ' the geographical longitude, and ρ a factor, varying with the latitude, such that the radius of the Earth at latitude φ is $a\rho$.

In the course of their investigations respecting nutation, Bessel, Peters, and Nyrén have made use of pendulum experiments to find the values of certain functions which they called P and P′. The relation between l, P, and P′ is*

$$l = 1 + P\sin^2\varphi' + P'\left(\sin^3\varphi' - \frac{3}{5}\sin\varphi'\right) \qquad (417)$$

where the length of the pendulum at the equator is taken as unity. With that unit, formula (9) gives

$$l = 1 + \left(\frac{10a}{l_0 t_1^2} - \varepsilon\right)\sin^2\varphi \qquad (418)$$

and by equating (417) and (418)

$$\left(\frac{10a}{l_0 t_1^2} - \varepsilon\right)\sin^2\varphi = P\sin^2\varphi' + P'\left(\sin^3\varphi' - \frac{3}{5}\sin\varphi'\right) \qquad (419)$$

At the poles $\varphi = \varphi'$, and as all powers of the sine are there unity, (419) reduces to

$$\varepsilon = \frac{10a}{l_0 t_1^2} - P \mp \frac{2}{5}P' \qquad (420)$$

or, with our adjusted values of a, l_0 and t_1

$$\varepsilon = 0\cdot008\,669\,52 - P \mp 0\cdot4P' \qquad (421)$$

in which the double sign is to be taken negative for the northern, and positive for the southern hemisphere of the Earth. To distinguish the two values of ε thus arising, we shall call the former ε', and the latter ε''.

In 1818, from pendulum experiments at 31 stations, Bessel found†

$$P = +0\cdot005\,444\,8 \qquad P' = +0\cdot000\,668\,9$$

whence, by (421)

$$\varepsilon' = 1 : 338\cdot17 \qquad \varepsilon'' = 1 : 286\cdot34 \qquad \tfrac{1}{2}(\varepsilon' + \varepsilon'') = 1 : 310\cdot11$$

In 1841, from experiments at 54 stations, Dr. C. A. F. Peters found‡

$$P = +0\cdot005\,233 \qquad P' = -0\cdot000\,334$$

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* 1, p. 130. † 1, p. 131. ‡ 109, p. 170.

whence

$$\varepsilon' = 1 : 280{\cdot}10 \qquad \varepsilon'' = 1 : 302{\cdot}76 \qquad \tfrac{1}{2}(\varepsilon' + \varepsilon'') = 1 : 290{\cdot}99$$

In 1872, from experiments at 74 stations, NYRÉN found*

$$P = +0{\cdot}005\,194 \qquad P' = -0{\cdot}000\,134$$

whence

$$\varepsilon' = 1 : 283{\cdot}36 \qquad \varepsilon'' = 1 : 292{\cdot}23 \qquad \tfrac{1}{2}(\varepsilon' + \varepsilon'') = 1 : 287{\cdot}73$$

All these values of the Earth's flattening should have been included in Table XI.

The fifth volume of the "Account of the operations of the great trigonometrical survey of India" contains a vast mass of data respecting pendulum experiments, but it was impossible to utilize them in the present investigation, because no general results are given, either for the length of the seconds pendulum or for the figure of the Earth, and none can be deduced without a large expenditure of labor.

In 1890 Professor NEWCOMB rediscussed the observations of the transits of Venus which occurred in 1761 and 1769, and found from them, for the solar parallax, $8{\cdot}79'' \pm 0{\cdot}034''$.†

30.—SUMMARY OF RESULTS.

Three essentially different systems of astronomical constants are given in the preceding pages, namely, (*a*) in Table VI, a system based upon General CLARKE'S spheroid of 1880, whose semiaxes are specified by the equations (3); (*b*), on page 111, a system based upon CLARKE'S value of the Earth's equatorial semidiameter, to wit, 20 926 202 feet, and adapted to any possible value of the flattening; and (*c*), in Table XVIII, a system in which the size and figure of the Earth were included among the quantities determined by the general adjustment. The latter system is certainly the most probable of the three, and for convenience of reference the values of all the quantities involved in it are here collected and appended. They are to be regarded as the definitive results of the investigations embraced in this paper.

Size, Figure, Density, and Moments of Inertia of the Earth.

Equatorial semidiameter	$= a =$	$20\,925\,293 \pm 409{\cdot}4$ feet
	$=$	$3\,963{\cdot}124 \pm 0{\cdot}078$ miles
	$=$	$6\,377\,972 \pm 124{\cdot}8$ meters.
Polar semidiameter	$= b =$	$20\,855\,590 \pm 325{\cdot}1$ feet
	$=$	$3\,949{\cdot}922 \pm 0{\cdot}062$ miles
	$=$	$6\,356\,727 \pm 99{\cdot}09$ meters.
One Earth quadrant	$=$	$393\,775\,819 \pm 4\,927$ inches,
	$=$	$32\,814\,652 \pm 410{\cdot}6$ feet,
	$=$	$6\,214{\cdot}896 \pm 0{\cdot}078$ miles,
	$=$	$10\,001\,816 \pm 125{\cdot}1$ meters.

$$\text{Flattening} = \frac{a-b}{a} = \frac{1}{300{\cdot}205 \pm 2{\cdot}964}$$

$$\text{Eccentricity} = \frac{a^2-b^2}{a^2} = 0{\cdot}006\,651\,018$$

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*104, p. 57. †234½, p. 402.

$$\text{Log. } \rho = 9.999\,2772\,758 + 0.000\,7245\,325 \cos 2\varphi$$
$$- 0.000\,0018\,131 \cos 4\varphi$$
$$+ 0.000\,0000\,047 \cos 6\varphi$$
$$\varphi - \varphi' = 688.2242'' \sin 2\varphi - 1.1482'' \sin 4\varphi + 0.0026'' \sin 6\varphi$$

Mean density of the Earth $= 5.576 \pm 0.016$

Surface density of the Earth $= 2.56 \pm 0.16$

Moments of inertia of the earth, $(C - A) : C = 0.003\,265\,21 = 1 : 306.259$

$$C - A = 0.001\,064\,767\ E'a^2$$
$$A = B = 0.325\,029\ E'a^2$$
$$C = 0.326\,094\ E'a^2$$

Length of the Seconds Pendulum.

$$l = 39.012\,540 + 0.208\,268 \sin^2\varphi \text{ inches,}$$
$$= 3.251\,045 + 0.017\,356 \sin^2\varphi \text{ feet,}$$
$$= 0.990\,910 + 0.005\,290 \sin^2\varphi \text{ meter.}$$

Acceleration by Gravity, per Second of Mean Solar Time.

$$g = 32.086\,528 + 0.171\,293 \sin^2\varphi \text{ feet,}$$
$$= 9.779\,886 + 0.052\,210 \sin^2\varphi \text{ meters.}$$

Length of the Year.

Sidereal year $= 365.256\,357\,8$ mean solar days,

$= 365^d\ 06^h\ 09^m\ 09.314^s$

$= 31\,558\,149.314$ mean solar seconds

Tropical year $= 365.242\,199\,870^d - 0.000\,006\,212\,4^d\left(\dfrac{t - 1850}{100}\right)$

$= 365^d\ 05^h\ 48^m\ 46.069^s - 0.536\,75^s\left(\dfrac{t - 1850}{100}\right)$

Length of the Month.

Sidereal month $= 27.321\,661\,162^d - 0.000\,000\,262\,40^d\left(\dfrac{t - 1800}{100}\right)$

$= 27^d\ 07^h\ 43^m\ 11.524^s - 0.022\,671^s\left(\dfrac{t - 1800}{100}\right)$

Synodical month $= 29.530\,588\,435^d - 0.000\,000\,306\,96^d\left(\dfrac{t - 1800}{100}\right)$

$= 29^d\ 12^h\ 44^m\ 02.841^s - 0.026\,522^s\left(\dfrac{t - 1800}{100}\right)$

Length of the Sidereal Day.

86 164.099 65 mean solar seconds.

Ratio of the Mean Motions of the Sun and Moon.

$$m = 0.074\,801\,329\,112$$

Masses of the Planets.

N. B.—The value given for the Earth is the combined mass of the Earth and Moon.

Mercury = 1 : (8 374 672 ± 1 765 762)
Venus = 1 : (408 968 ± 1 874)
Earth = 1 : (327 214 ± 624)
Mars = 1 : (3 093 500 ± 3 295)
Jupiter = 1 : (1 047·55 ± 0·20)
Saturn = 1 : (3 501·6 ± 0·78)
Uranus = 1 : (22 600 ± 36)
Neptune = 1 : (18 780 ± 300)
Moon = 1 : (81·068 ± 0·238)

Solar Parallax.

$$p = 8{\cdot}809\,05'' \pm 0{\cdot}005\,67''$$

Mean distance from Earth to Sun = 92 796 950 ± 59 715 miles,
= 149 340 870 ± 96 101 kilometers.

Eccentricity of the Earth's Orbit.

$$e_1 = 0{\cdot}016\,771\,049 - 0{\cdot}000\,000\,424\,5\,(t - 1850) - 0{\cdot}000\,000\,001\,367\left(\frac{t-1850}{100}\right)^2$$

Lunar Inequality of the Earth.

$$\mathrm{L} = 6{\cdot}522\,94'' \pm 0{\cdot}018\,54''$$

Lunar Parallax.

$$\mathrm{P} = 3\,422{\cdot}542\,16'' \pm 0{\cdot}125\,33''$$
$$\sin \mathrm{P} : \mathrm{arc}\ 1'' = 3\,422{\cdot}385\,11'' \pm 0{\cdot}125\,33''$$

Mean distance from Earth to Moon = 60·269 315 ± 0·002 502 terrestrial radii,
= 238 854·75 ± 9·916 miles,
= 384 396·01 ± 15·958 kilometers.

Eccentricity and Inclination of the Moon's Orbit.

$$e_2 = 0{\cdot}054\,899\,720$$
Delaunay's $\gamma = \sin \tfrac{1}{2}\,\mathrm{I} = 0{\cdot}044\,886\,793$
$$\mathrm{I} = 5^\circ\ 08'\ 43{\cdot}3546''$$

Mean Motion of the Moon's Node in 365¼ Days.

$$\mu = -\,19^\circ\ 21'\ 19{\cdot}6191'' + 0{\cdot}141\,36''\left(\frac{t-1800}{100}\right)$$

Parallactic Inequality of the Moon.

$$Q = 124{\cdot}951\,26'' \pm 0{\cdot}081\,97''$$

Secular Part of the Precession.

$$\varkappa = 17{\cdot}330\,968'' \qquad \varepsilon = 2{\cdot}183\,513$$

$$\psi = (50{\cdot}357\,10'' \pm 0{\cdot}003\,49'')(t - 1850) - 0{\cdot}000\,106\,70''\,(t - 1850)^2$$

$$\omega = 23^\circ\,27'\,31{\cdot}47'' \pm 0{\cdot}345'' + 0{\cdot}000\,006\,41''\,(t - 1850)^2$$

$$\psi_1 = (50{\cdot}236\,15'' \pm 0{\cdot}003\,49'')(t - 1850) + 0{\cdot}000\,110\,23''\,(t - 1850)^2$$

$$\omega_1 = 23^\circ\,27'\,31{\cdot}47'' \pm 0{\cdot}345 - 0{\cdot}466\,57''\,(t - 1850) - 0{\cdot}000\,000\,73''\,(t - 1850)^2$$

Constant of Nutation.

$$\mathfrak{A} = 9{\cdot}220\,54'' \pm 0{\cdot}008\,59'' + 0{\cdot}000\,009\,04''\,(t - 1850$$

General Formulæ for Precession and Nutation.

[For explanation, see page 86.]

$$\psi = [50\,357\,10'' + 0{\cdot}000\,049\,43''\,(t_0 - 1850)]\,(t - t_0) - 0{\cdot}000\,106\,70''\,(t - t_0)^2 + \Psi$$

$$\omega = 23^\circ\,27'\,31{\cdot}47'' - 0{\cdot}466\,57''\,(t_0 - 1850) - 0{\cdot}000\,000\,73''\,(t_0 - 1850)^2 + 0\,000\,006\,41''\,(t - t_0)^2 + \Omega$$

$$\lambda = [0{\cdot}131\,84'' - 0{\cdot}000\,186\,56''\,(t_0 - 1850)](t - t_0) - 0{\cdot}000\,236\,53''\,(t - t_0)^2$$

$$m = 46{\cdot}063\,08'' + 0{\cdot}000\,277\,25''\,[(t_0 - 1850) + (t - t_0)]$$

$$n = 20{\cdot}046\,58'' - 0{\cdot}000\,084\,94''\,[(t_0 - 1850) + (t - t_0)]$$

$$\psi_1 = [50{\cdot}236\,15'' + 0{\cdot}000\,220\,45''\,(t_0 - 1850)](t - t_0) + 0{\cdot}000\,110\,23''\,(t - t_0)^2 + \Psi$$

$$\omega_1 = 23^\circ\,27'\,31{\cdot}47'' - 0{\cdot}466\,57''\,(t_0 - 1850) - 0{\cdot}000\,000\,73''\,(t_0 - 1850)^2 - [0{\cdot}466\,57'' + 0{\cdot}000\,001\,46''\,(t_0 - 1850)](t - t_0) - 0{\cdot}000\,000\,73''\,(t - t_0)^2 + \Omega$$

$$\varphi'' = [0{\cdot}469\,51'' - 0{\cdot}000\,006\,89''\,(t_0 - 1850)](t - t_0) - 0{\cdot}000\,003\,45''\,(t - t_0)^2$$

$$\theta'' = 173^\circ\,34'\,54'' + 32{\cdot}655''\,(t_0 - 1850) - 8{\cdot}791''\,(t - t_0)$$

$$M = 173^\circ\,34'\,54'' + 32{\cdot}655''\,[(t_0 - 1850) + (t - t_0)]$$

$$\begin{aligned}\Psi = &- [17{\cdot}2463'' + 0{\cdot}000\,173\,2''\,(t - 1850)]\sin\Omega \\ &+ [0{\cdot}2070'' + 0{\cdot}000\,000\,2''\,(t - 1850)]\sin 2\Omega \\ &- [1{\cdot}2642'' + 0{\cdot}000\,001\,2''\,(t - 1850)]\sin 2\odot \\ &- [0{\cdot}2043'' + 0{\cdot}000\,000\,2''\,(t - 1850)]\sin 2☾ \\ &+ [0{\cdot}1273'' + 0{\cdot}000\,000\,1''\,(t - 1850)]\sin A_\odot \\ &+ [0{\cdot}0686'' + 0{\cdot}000\,000\,1''\,(t - 1850)]\sin A_☾\end{aligned}$$

$$\begin{aligned}\Omega = &+ [9{\cdot}2205'' + 0{\cdot}000\,009\,0''\,(t - 1850)]\cos\Omega \\ &- [0{\cdot}0899'' - 0{\cdot}000\,000\,5''\,(t - 1850)]\cos 2\Omega \\ &+ [0{\cdot}5486'' - 0{\cdot}000\,002\,9''\,(t - 1850)]\cos 2\odot \\ &+ [0{\cdot}0886'' - 0{\cdot}000\,000\,5''\,(t - 1850)]\cos 2☾\end{aligned}$$

Constant of Aberration.

$$\alpha = 20.454\,51'' \pm 0.012\,58''$$

Light Equation and Velocity of Light.

$$\theta = 498.005\,95^s \pm 0.308\,34^s$$
$$V = 186\,337.00 \pm 49.722 \text{ miles per second,}$$
$$= 299\,877.64 \pm 80.019 \text{ kilometers per second.}$$

31.—CONCLUDING REMARKS.

The preceding section contains a thoroughly homogeneous system of constants, based upon an enormous mass of astronomical, geodetic, gravitational, and tidal observations which have required more than two hundred years for their accumulation, and now it only remains to inquire how these constants can be most readily improved in the future. The first step in that direction will be to examine the sources of their probable errors, all of which are exhibited in Table XVII, and in doing so it must not be forgotten that a quantity may contribute little to the probable errors in the table, either because its own probable error is small, or because it is very large. In the former case the uncertainty which it introduces will be intrinsically small, while in the latter it will be rendered small by its small coefficient, and these two cases must be carefully distinguished.

Table XVII shows that the probable error of the Earth's flattening arises quite equably from the probable errors of the observed values of P and ε, and these are almost its only sources.

Respecting P, we remark that as yet its value depends mainly upon meridian observations made at Greenwich and the Cape of Good Hope, but a value determined from observations made at some station near the equator is urgently needed to aid in ascertaining the true flattening of the Earth. Such a value could be obtained in a few months by employing the method of vertical transits to compare the position of the Moon when rising and setting with that of a neighboring star, and the work might well be undertaken by a small astronomical expedition. If, in addition to that, the observatories at the Cape of Good Hope, Melbourne, Sydney, Adelaide, Cordoba, and Santiago de Chile would co-operate with those of the northern hemisphere in systematically making meridian observations of the Moon for two or three years, the resulting data would doubtless lead to a material improvement of our knowledge concerning the lunar parallax and the flattening of the Earth. A probable error of $\pm 0.026\,89''$ in each of the two values of P, found respectively for the equator and for the pole, would give rise to an uncertainty of one unit in the reciprocal of ε.

On page 126 the observed value of ε was found from the four values given respectively by the geodetic arcs, HELMERT's reduction of the pendulum experiments, precession and nutation, and perturbations of the Moon; but now we must also take account of the last general adjustment. With the values of $\mathfrak{P}$ and $\mathfrak{A}$ from Table XVIII, formula (288) gives $(C - A) / C = 0.003\,265\,29$, whence, by Table IX, $\varepsilon = 1 : 297.91$. Further, although in Table XII HANSEN's values of the Moon's perturbations give $\varepsilon = 1 : 294{\cdot}93$, AIRY's value, founded upon much better known data, gives $\varepsilon = 1 : 311{\cdot}40$, and there is no way of determining what the true value is. In short, the general adjustment, the pendulum experiments, and precession and nutation give a flattening differing little from 1 : 300, the result from lunar perturbations is uncertain within rather wide limits, and the geodetic arcs give 1 : 293·5. Thus it appears that the geodetic value stands quite alone, and as it is almost certainly erroneous, the probable error of the observed value of ε could be largely diminished by making ε depend solely upon the results from pendulum experiments and precession and nutation. At the present time more pendulum experiments, both in the neighborhood of the equator, and as near as possible to the poles, are among the things most needed to perfect our knowledge of the figure of the Earth. Experiments in middle latitudes will contribute little to that end, however useful they may be for other purposes.

Unfortunately no value of the Earth's flattening can be deduced, either from the pendulum experiments or from precession and nutation, without making some assumption respecting the unknown constitution of the interior of the Earth. Notwithstanding the difficulties which arise in connection with the rigidity exhibited by the Earth under the action of the forces which generate precession, nutation, and the tides, the theory of a comparatively thin crust resting in approximate hydrostatic equilibrium upon a denser substratum is favored by enough facts to render it very plausible. If adopted, it necessarily entails the use of the condensation method for the reduction of the pendulum experiments, but it is not clear that HELMERT's numerical result depends essentially upon that method, because UNFERDINGER got nearly the same result by means of a totally different theory. Furthermore, if the interior of the Earth ever was liquid, LE GENDRE's law of the distribution of density therein must be at least approximately true, and some evidence to that effect is afforded by the substantial agreement between the values of ε found respectively from the general adjustment, the pendulum experiments, and precession and nutation.

As stated on page 5, the geodetic arcs used in determining the figure of the Earth are located in England, France, Russia, India, South Africa, and Peru. Hitherto no great arcs except those of the Lake Survey* have been measured in the United States, but several have been projected by the U. S. Coast and Geodetic Survey, and some of these are so far advanced that their completion may be expected in the near future. All who comprehend the importance of the subject hope that the United States will not long remain behind other great nations in contributing her share toward determining the size and figure of the globe we inhabit.

The extent to which the flattening of the Earth deduced from the geodetic arcs differs from that found by other trustworthy methods seems to indicate that the existing arcs are not well adapted to determine the relative lengths of the polar and equatorial semidiameters. Possibly the difficulty may arise from the preponderating influence which the ten southern degrees of the Indian arc now have in determining the length of the polar semidiameter,† and if so, a meridianal arc situated within the

* The Lake Survey arcs were not published by Gen. COMSTOCK until two years after CLARKE's spheroid of 1880. The arc of parallel is in latitude 42°, and therefore very favorably situated for determining ε.

† 13, pp. 310–311.

United States, and having its center about latitude 35°, would be of great value. To determine the reciprocal of ε with a probable error not exceeding one unit, the probable error of each of the semidiameters must be reduced to ± 164·4 feet. With our present resources it will probably be best to find the value of ε from pendulum experiments, precession, nutation, and perturbations of the Moon, and after the ratio of the Earth's semidiameters has thus become known, their absolute lengths can be got from the arcs with great exactness.

Table XX has been compiled from Table XVII in order to show as clearly as possible the relative importance of each quantity in determining the probable error of every one of the twelve quantities included in the general adjustment. The several quantities are entered symbolically in the first column of the table, and on the same line with each of them the principal sources of its probable error are indicated in the order of their importance by the numerals 1, 2, 3, etc. For example, in determining the probable error of Q, the parallactic inequality of the Moon's motion, the quantities whose observed values have the most influence are, 1, the constant of aberration; 2, the solar parallax; 3, the velocity of light; 4, the mass of the Earth; and, 5, the light equation. A glance at the table shows that the numeral 1 occurs once in each of the columns P, 𝔓, V, and ε, three times in the column L, and five times in the column α. It is therefore evident that in order to improve the system of constants determined in

TABLE XX.—*Origin of the Probable Errors of the Adjusted Quantities.*

Quantities.	Principal sources of the Probable Errors.											
	p	P	𝔓	𝔑	Q	L	a	θ	V	E	M	ε
p	2	. .	. .	. .	. .	. .	1	. .	3	4	. .	. .
P	. .	1	. .	. .	. .	3	. .	. .	. .	. .	. .	2
𝔓	. .	. .	1	. .	. .	. .	. .	. .	. .	. .	. .	. .
𝔑	. .	. .	. .	2	. .	1	. .	. .	. .	. .	. .	. .
Q	2	. .	. .	. .	. .	. .	1	5	3	4	. .	. .
L	. .	3	. .	2	. .	1	. .	. .	. .	. .	4	. .
a	2	. .	. .	. .	. .	. .	1	4	. .	3	. .	. .
θ	2	. .	. .	. .	. .	. .	1	4	. .	3	. .	. .
V	. .	. .	. .	. .	. .	. .	. .	. .	1	. .	. .	. .
E	2	. .	. .	. .	. .	. .	1	. .	3	4	. .	. .
M	. .	3	. .	2	. .	1	. .	. .	. .	. .	. .	. .
ε	. .	2	. .	4	. .	3	. .	. .	. .	. .	. .	1

the present paper the quantities whose observed values we should endeavor to improve are, 1, the constant of aberration; 2, the lunar inequality of the Earth's motion; 3, the solar parallax; and, 4, the constant of nutation. The constant of precession and the velocity of light are so well determined as to be virtually independent of all other quantities, while the parallactic inequality of the Moon and the mass of the Moon from the tides are so uncertain that they have little influence upon anything.

On account of the great importance of the constant of aberration in our adjustment, it seems desirable to recall the fact that at present we have no satisfactory theory of aberration. The constant in question has long been regarded as the ratio of the Earth's mean orbital velocity to the velocity of light, but notwithstanding the plausibility of that assumption it has never been rigorously deduced from the undulatory theory of light.

The desiderata for the improvement of the system of constants discussed in the present paper may now be recapitulated as follows:

1. The parallax of the Moon should be determined by the diurnal method, at one or more stations as near as possible to the equator.

2. The observatories in the northern and southern hemispheres should co-operate with each other for two or three years in systematically making meridian observations of the Moon to improve our knowledge of its parallax.

3. Pendulum experiments should be made at a large number of stations, located partly in the neighborhood of the equator and partly as near as possible to the poles. Experiments in middle latitudes are also desirable, but somewhat less necessary.

4. New determinations of the constants of aberration and nutation should be made by as many different methods as possible.

5. The meridian observations of the Sun, accumulated at the Greenwich and Washington Observatories during the last fifty years, should be discussed in such a way as to deduce from them the most probable coefficient of the lunar inequality of the Earth's motion.

6. New determinations of the solar parallax should be made by observing Mars during its opposition in 1892, and also by observing such asteroids as may come into favorable positions for that purpose.

7. The measurement of some of the great arcs included in the scheme of the U. S. Coast and Geodetic Survey should be completed as soon as possible.

There can be no doubt that the observations specified in paragraphs 1, 2, and 3 would materially improve our knowledge, both of the lunar parallax and of the Earth's flattening; but the probable errors of the constant of aberration, the constant of nutation, the coefficient of the lunar inequality of the Earth's motion, and the solar parallax are already so small that it will be exceedingly difficult to reduce them any further. Nevertheless, the attempt should be made.

The values given on page 139 for the Earth's moments of inertia have been derived from the adjusted values of $\mathfrak{P}$, $\mathfrak{N}$, a, l, and ε by means of the formulæ (288) and (306). These moments of inertia determine the period of the Eulerian nutation; and if there is any want of coincidence between the Earth's axis of rotation and its principal axis of inertia, they also determine the period of the resulting variations of latitude. Although arising from widely different causes, these two periods depend upon the same function of the moments of inertia, and are therefore alike, the length of each being A / (C — A) sidereal days,* or

$$\frac{A}{C - A} \times \frac{365{\cdot}2564}{366{\cdot}2564} = 0{\cdot}997\ 270 \frac{A}{C - A} \text{ mean solar days}$$

With our values of A and C, the numerical results are 305·26 sidereal, or 304·42 mean solar days.

* 109½, p. 126, 103½, p. 569, and 104, p. 18.

The computations in the preceding pages were all made by the author, and as it proved impracticable to have them checked by any other person, certain parts of the work were done in duplicate by means of two different sets of formulæ, and special care was taken to apply check equations wherever possible. Most of the numerical operations were effected upon a THOMAS arithmometer capable of dealing with factors having ten or eleven significant figures, and advantage was taken of that circumstance to increase the accuracy of the checks by carrying the results somewhat beyond the usual number of decimals. The accompanying probable errors will suffice to prevent these decimals from creating any false impression respecting the degree of accuracy attained.

In conclusion the author desires to thank his colleagues, Professors ASAPH HALL and EDGAR FRISBY, U. S. Navy, and his friend, Mr. R. S. WOODWARD, U. S. Coast and Geodetic Survey, for the interest they have taken in the preparation of the present paper, for their kindness in examining much of it, both in manuscript and in proof, and for many valuable criticisms and suggestions.

32.—BIBLIOGRAPHY.

The titles of most of the works consulted in the preparation of the present paper are given in the following list, where each is distinguished by a reference number. For the sake of brevity these numbers have been used instead of the titles themselves throughout the preceding pages, and thus all citations of authorities have been reduced to a number and a page. For example, the first citation occurs on page 3, and is in the form "13, p. 157", which signifies page 157 of the work whose title is number 13, namely, Colonel A. R. CLARKE's Geodesy.

The following abbreviations are employed:

Ast. Nach. *for* Astronomische Nachrichten, begründet von H. C. Schumacher (Altona und Kiel).

Comptes Rendus *for* Comptes Rendus hebdomadaires des séances de l'Académie des Sciences (Paris).

L., E. and D. Phil. Mag. *for* London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science. Being a continuation of Tilloch's "Philosophical Magazine," Nicholson's "Journal," and Thomson's "Annals of Philosophy" (London).

Mem. Roy. Ast. Soc. *for* Memoirs of the Royal Astronomical Society (London).

Month. Not. *for* Monthly Notices of the Royal Astronomical Society (London), containing papers, abstracts of papers, and reports of the proceedings of the Society.

Phil. Trans. *for* Philosophical Transactions of the Royal Society of London.

Proc. Roy. Soc. *for* Proceedings of the Royal Society of London.

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